ES.1803 Practice Questions – Final Quiz, Spring 2024

Important: Not every topic is covered here. When preparing for the final be sure to look over other review materials as well as old psets and exams.

On the final quiz, you will be given the formula posted alongside these problems.

Problem 1. For the DE $\dfrac{dy}{dx} = -\dfrac{y}{x} + 3\,x$:

(a) Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes -1 and 0.

(b) For the solution curve passing through the point (1,2): If Euler's method with step-size $h = 0.1$ were used to approximate $y(1.1)$, would the approximation come out too high or too low? Explain.

(c) For the solution with $y(1) = 2$, compute the Euler approximation to $y(1.1)$ using step-size $h = 0.1$.

(d) The functions $y_1 = x^2$ and $y_2 = x^2 + \frac{1}{x}$ are solutions to this DE. If $y = y(x)$ is the solution satisfying the IC $y(1) = 1.5$, show that $100 \leq y(10) \leq 100.1$. Do we need to include the equal signs in this inequality? Why or why not?

(e) Find the general solution the DE and verify the prediction of Part (b).

Problem 2. Let $P(D) = D^2 + bD + 5I$ where $D = \frac{d}{dt}$.

(a) For what range of the values of $b \geq 0$ will the solutions to $P(D)y = 0$ exhibit oscillatory behavior?

(b) For $b = 4$, solve the DEs (i) $P(D)y = 4\,e^{2t}\sin(t)$ (ii) $P(D)y = 4\,e^{2t}\cos(t)$

Write your answers in both amplitude-phase and rectangular form.

(c) Given $b = 2$, for what ω does $P(D)y = \cos(\omega t)$ have the biggest response?

Problem 3. Find the general solution to the DE $(D^3 - I)\,y = e^x$

Express the answer using *real*-valued functions only.

Problem 4. Let L denote the differential operator $Ly = D^2y - \frac{1}{x}\,Dy + 4x^2y$, where $D = \frac{d}{dx}$.

(a) Show that the DE $Ly = 0$ has solutions $y_1(x) = \cos\left(x^2\right)$ and $y_2(x) = \sin\left(x^2\right)$.

(b) Show that the initial value problem

$$Ly = 0, \quad y(0) = 0,\; y'(0) = 0$$

has *more than one* solution. Why doesn't this contradict the Existence and Uniqueness Theorem? On what intervals does existence and uniqueness hold?

Problem 5. Suppose that a population of variable size (in some suitable units) $p(t)$ follows the growth law $\dfrac{dp}{dt} = p^3 - 4p^2 + 4p$. Without solving the DE explicitly:

(a) Find all critical points and classify each according to its stability type using a phase line diagram.

(b) Draw a rough sketch (on p-vs.-t axes) of the family of solutions. What happens to the population in the long-run if it starts out at size 1 unit; at size 3 units ?

(c) Explain why the rate equation given by the DE was all we needed to get the answer to Part (b).

(d) Now we'll add a harvesting parameter to the system: $p' = p^3 - 4p^2 + 4p - r$.

(i) Draw the bifurcation diagram for this system.

(ii) Give the bifurcation points.

(iii) For what values of r is the population sustainable?

Problem 6. (a) Solve $2y'' - 2y' - 4y = \delta(t)$, with rest IC.

(b) Solve $2y'' + 2y = \delta(t-3)$, with rest IC.

(c) Solve $x' + tx = \delta(t-5)$

Problem 7. Let $f(t)$ be 0 for $t < 0$ and $3e^{-2t}$ for $t > 0$. Compute the generalized derivative of $f(t)$.

Problem 8. For $f(t) = t$ on $0 < t < 1$:

(a) Sketch the following periodic extensions of f over three or more full periods in the cases.

(i) Even period 2 extension (ii) Odd period 2 extension (iii) Period 1 extension.

In all three cases chose endpoint values that show where the Fourier series expansion will converge. (Do this *without* computing the Fourier series).

(b) Compute the Fourier sine series of f.

(c) Find the periodic solution to the DE $x'' + 10x = \tilde{f}_{\text{odd}}(t)$. Does near-resonance occur in this situation? If so, which frequency in the 'driving force' $\tilde{f}_{\text{odd}}(t)$ produces it?

(d) Solve the DE $x' + 10x = \tilde{f}_{\text{odd}}(t)$.

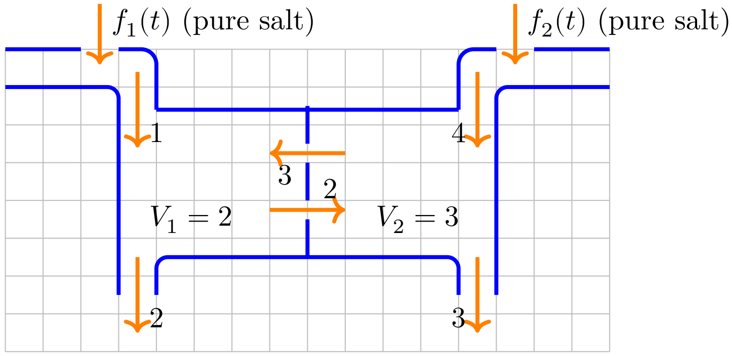
Problem 9. (a) Write down the wave equation with IC's and BC's for the string of length 1, with clamped ends, wave speed 2, initially at equilibrium, struck at time 0. Then derive the Fourier series solution using separation of variables.

(b) Give the explicit solution to the equation of Part (a) when the initial velocity is given by $f(x) = x$ on $0 < x < 1$ (as if that were possible!).

Problem 10. Find the general real-valued solution to the system of DEs:

$$x' = x - 2y, \quad y' = 4x + 3y.$$

Problem 11. Given the following two-tank mixing system with flow rates, inputs and volumes as shown. (All unit are compatible; $f_1(t)$ and $f_2(t)$ denote salt rates in $\frac{\text{mass}}{\text{time}}$.)



(a) Let x and y be the amount of salt in tanks 1 and 2 respectively. Set up a system of DEs modeling x, y.

(b) Suppose the input salt rates $f_1(t)$ and $f_2(t)$ are constant. Show that the system approaches a state in which the final concentrations are constant.

For more problems on linear and nonlinear systems see Practice Quiz 7, Problems 8-11.

End practice final