ES.1803 Practice 2 Final Quiz, Spring 2024 Solutions

This practice contains no linear algebra problems.

Problem 1.

(a) *Find* all *solutions to the DE* $\dfrac{dy}{dx} = y^2$.

Solution: We separate variables and also add any lost solutions.

Separable: $\dfrac{dy}{y^2} = dx \Rightarrow -\dfrac{1}{y} = x + C$

$\Rightarrow \boxed{y = -\dfrac{1}{x+C}.}$ Plus one lost solution: $\boxed{y = 0.}$

(b) *Give a definite integral solution to the following IVP*

$$y' + 4\,y = \cos(t^2 + t^3);\; y(0) = 2.$$

You do not have to evaluate the integral.

Solution: Linear first-order DE, unusual input: use the variation of parameters formula.

$y_h(t) = e^{-4t}$.

$$y(t) = y_h(t)\left(\int_0^t f(u)/y_h(u)\,du + y_0/y_h(0)\right) = \boxed{e^{-4t}\left(\int_0^t e^{4u}\cos(u^2+u^3)\,du + 2\right).}$$

Problem 2.

(a) *Give the general* real *solution to* $\ddot{x} + 3\dot{x} + 4x = e^{2t} + t + 3$.

Solution: Homogeneous equation: $\ddot{x} + 3\dot{x} + 4x = 0$.

Characteristic equation: $r^2 + 3r + 4 = 0 \Rightarrow r = \dfrac{-3 \pm \sqrt{9-16}}{2} = \dfrac{-3 \pm \sqrt{7}\,i}{2}$.

$\Rightarrow x_h(t) = c_1 e^{-3t/2}\cos(\sqrt{7}\,t/2) + c_2 e^{-3t/2}\sin(\sqrt{7}/2\,t)$.

Particular pieces:

$\ddot{x}_1 + 3\dot{x}_1 + 4x_1 = e^{2t}$: ERF $\Rightarrow x_1(t) = \dfrac{e^{2t}}{P(2)} = \dfrac{e^{2t}}{14}$.

$\ddot{x}_2 + 3\dot{x}_2 + 4x_2 = t + 3$: Try $x_2 = At + B$

Substitution $\Rightarrow 3(A) + 4(At + B) = t + 3 \quad \Rightarrow 4At + 3A + 4B = t + 3$

$\Rightarrow A = 1/4,\ B = 9/16 \quad \Rightarrow x_2(t) = \dfrac{t}{4} + \dfrac{9}{16}$.

$$\boxed{x(t) = x_1(t) + x_2(t) + x_h(t) = \frac{e^{2t}}{14} + \frac{t}{4} + \frac{9}{16} + c_1 e^{-3t/2}\cos\left(\frac{\sqrt{7}\,t}{2}\right) + c_2 e^{-3t/2}\sin\left(\frac{\sqrt{7}\,t}{2}\right).}$$

(b) *If the differential operator in Part (a) models a physical system is the system stable? Eplain how you know.*

Solution: Yes, the system is stable because the characteristic roots have negative real part.

(c) *What is the amplitude response for the system* $\ddot{x} + 3\dot{x} + 4x = \cos(\omega t)$*, where* $\cos(\omega t)$ *is the input?*

Solution: The SRF gives the periodic solution: $x_p(t) = \dfrac{\cos(\omega t - \phi)}{|P(i\omega)|}$. Since $\cos(\omega t)$ is the input, the gain is $g(\omega) = \dfrac{1}{|P(i\omega)|}$.

Computing: $P(i\omega) = 4 - \omega^2 + 3i\omega \Rightarrow \boxed{g(\omega) = \dfrac{1}{\sqrt{(4-\omega^2)^2 + 9\omega^2}}.}$

(d) *For what values of* k *does* $\ddot{x} + 3\dot{x} + kx = 0$ *have oscillatory solutions?*

Solution: Characteristic roots are $\dfrac{-3 \pm \sqrt{9-4k}}{2}$.

Oscillatory $\Leftrightarrow$ complex roots $\Leftrightarrow$ $9 - 4k < 0$ $\Leftrightarrow$ $\boxed{k > 9/4.}$

Problem 3. *Let* $P(D)$ *be a constant coefficient differential operator.*

Suppose that the DE $P(D)x = f$ *with* $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(nt)$ *has the periodic solution*

$$x_p(t) = \sum_{n=1}^{\infty} \frac{1}{|5-n^2|(n^2)}\sin(nt - \phi(n)), \quad \text{where } \phi(n) = \begin{cases} 0 & \text{for } n < \sqrt{5} \\ \pi & \text{for } n > \sqrt{5} \end{cases}.$$

(a) *Without finding* $P(D)$ *write down the periodic solution to the DE* $P(D)x = \sin(3t)$*.*

Solution: The solution $x_p(t)$ was found by superposition, so, stripping out the coefficients from the input, we see the solution to $P(D)x = \sin(3t)$ is $\boxed{x(t) = \dfrac{\sin(3t-\pi)}{|5-9|} = -\dfrac{1}{4}\sin(3t).}$

(b) *Find* $P(D)$*.*

Solution: Again, since $x_p(t)$ is found by superposition we see $|P(in)| = |5 - n^2|$ and $\text{Arg}(P(in)) = \phi(n)$. So, $P(D) = D^2 + 5I$.

Problem 4. *Solve* $x'' + 5x = 5\cos(\omega t)$*.* (Be sure to do this for every value of ω.)

Solution: Use the SRF: $x_p(t) = \dfrac{5\cos(\omega t - \phi)}{|P(i\omega)|}$ (as long as $P(i\omega) \neq 0$).

$$P(i\omega) = 5 - \omega^2; \quad |P(i\omega)| = |5 - \omega^2|; \quad \phi(\omega) = \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } \omega < \sqrt{5} \\ \pi & \text{if } \omega > \sqrt{5} \end{cases}$$

(We ignore $\omega = \sqrt{5}$ because then $P(i\omega) = 0$ and we need to use the Extended SRF.) So, for $\omega \neq \sqrt{5}$, we have $x_p(t) = \dfrac{5\cos(\omega t - \phi(\omega))}{|5-\omega^2|}$.

If $\omega = \sqrt{5}$ we need the ESRF: $x_p(t) = \dfrac{5t\cos(\omega t - \phi)}{|P'(i\omega)|}$, where $\phi = \text{Arg}(P'(i\omega))$.

$$P'(i\omega) = 2i\omega; \quad |P'(i\sqrt{5})| = 2\sqrt{5}; \quad \phi = \text{Arg}(P'(i\sqrt{5})) = \pi/2.$$

The full solution for the particular solution to the DE is

$$x_p(t) = \begin{cases} \dfrac{5\cos(\omega t)}{5-\omega^2} & \text{if } \omega < \sqrt{5} \\ \dfrac{5\cos(\omega t - \pi)}{|5-\omega^2|} = \dfrac{-5\cos(\omega t)}{|5-\omega^2|} & \text{if } \omega > \sqrt{5} \\ \dfrac{5t\cos(\sqrt{5}\,t - \pi/2)}{2\sqrt{5}} = \dfrac{5t\sin(\omega t)}{2\sqrt{5}} & \text{if } \omega = \sqrt{5}. \end{cases}$$

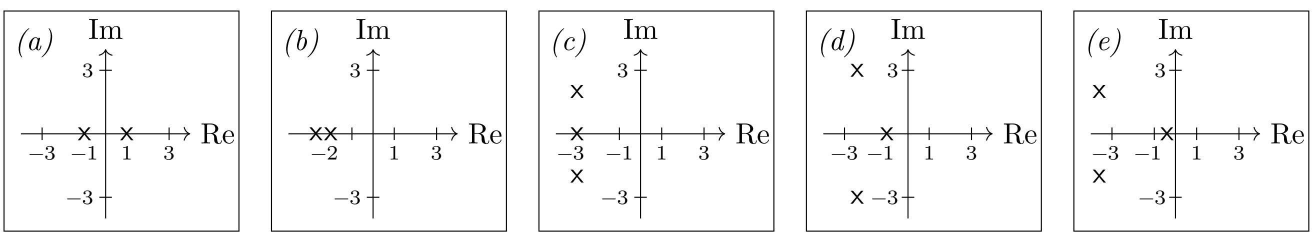
Problem 5.
Solve $(D^3 + I)y = 0.$

Solution: Characteristic equation $r^3 + 1 = 0$. So, $r^3 = -1 = e^{i(\pi + 2n\pi)}$. Thus,

$$r = e^{i(\pi/3 + 2n\pi/3)} = e^{i\pi/3},\ e^{i3\pi/3},\ e^{i5\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -1,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

So, $\boxed{y(t) = c_1 e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + c_2 e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right) + c_3 e^{-t}.}$

Problem 6.
Consider the following pole diagrams for 5 linear time invariant systems of the form $P(D)y = f$. (The diagrams are in the complex s-plane, where we consider the characteristic polynomial $P(s)$ to be a function of s.)



(a) *List all the stable systems.*

Solution: Systems are stable if all poles are in the left half-plane: $\boxed{\text{b, c, d, e}}$

(b) *Choose the stable system where the transient has the fastest decay.*

Solution: The decay rate is determined by the right-most pole.

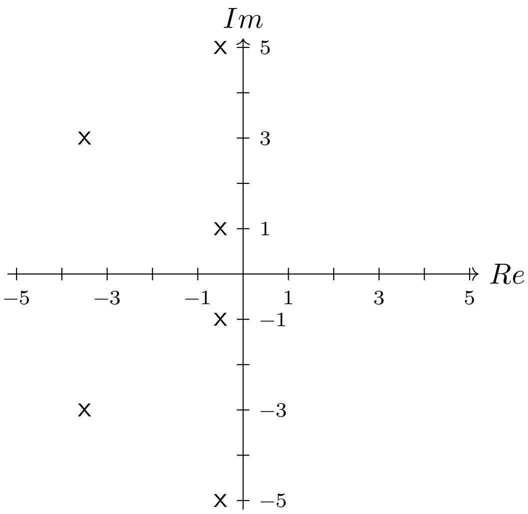
So we want the system with the right-most pole farthest to the left (i.e., the left-most right-most pole). $\Rightarrow$ $\boxed{\text{choose system (c).}}$

(c) *Choose the stable system where the transient decays as fast as possible without oscillation.*

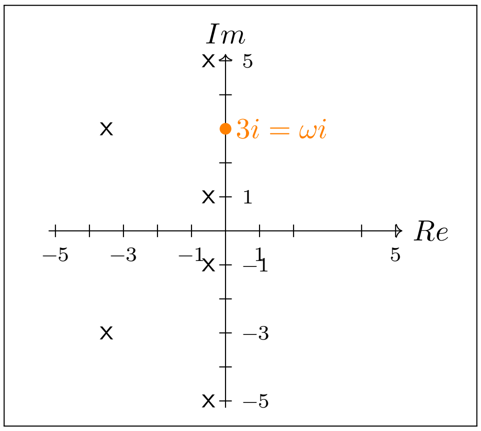
Solution: We only consider systems with all real poles. Among these we want the one with the right-most pole farthest to the left $\Rightarrow$ $\boxed{\text{choose system (b).}}$

(d) *Below is the pole diagram for a linear time invariant system with cosine input: $P(D)x = F_0\cos(\omega t)$. In this particular system ω must be an integer between 1 and 5 and it is critical that the response be kept as* small *as possible.*

What frequency ω would you use? Mark the pole diagram with a solid dot in the appropriate place to indicate this frequency.



Solution: The amplitude of the response to $\cos(\omega t)$ is given by $1/|P(i\omega)|$. Since the poles are where $1/|P(s)|$ goes to infinity, we want $i\omega$ to be as far as possible from the nearest pole. That is, we want ω to be 3. It should be marked on the imaginary axis in the diagram.



Problem 7.
For a linear, time invariant system, if the input is e^{st}, then the output is $G(s)e^{st}$. The gain factor $G(s)$ is called the complex gain of the system.

Suppose such a system has complex gain $G(s) = \dfrac{s}{(s^2+9)(s+7)(s+1)}$.

(a) *For what value of $\omega > 0$ will the input $\cos(\omega t)$ give the biggest response.*

Solution: Since the response to $e^{i\omega t}$ is $G(i\omega)e^{i\omega t}$, the amplitude of the response to $\cos(\omega t)$ is

$$|G(i\omega)| = \left|\frac{i\omega}{((i\omega)^2+9)(i\omega+7)(i\omega+1)}\right| = \left|\frac{\omega}{(9-\omega^2)(i\omega+7)(i\omega+1)}\right|.$$

Since the denominator is 0 when $\omega = 3$ the maximum amplitude is ∞ when $\boxed{\omega = 3}$. (That is, there is pure resonance at $\omega = 3$.)

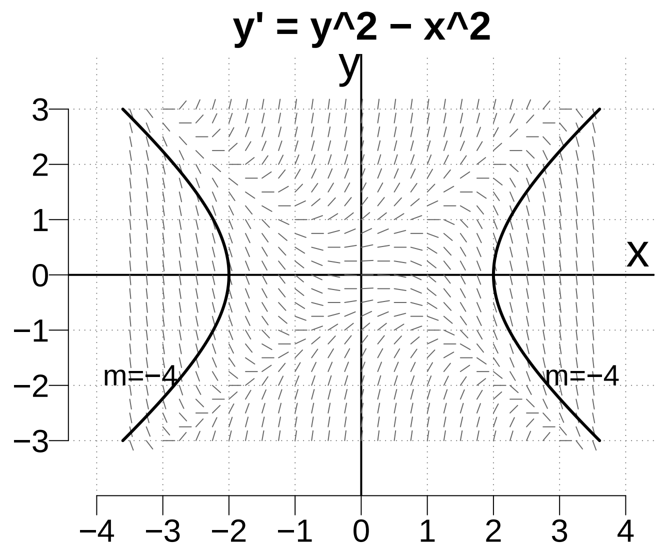
(b) *The system with input f satisfies the DE $P(D)y = f'$. What is $P(D)$?*

Solution: Letting $f = e^{st}$ the equation becomes $P(D)y = se^{st}$.

The ERF gives $y_p(t) = \dfrac{se^{st}}{P(s)}$. That is, $G(s) = \frac{s}{P(s)}$. Thus, $P(s) = (s^2+9)(s+7)(s+1)$.

So, $\boxed{P(D) = (D^2+9)(D+7)(D+1)}$.

Problem 8.
The DE for this problem is $y' = y^2 - x^2$. The direction field for this DE is shown. We also show the isocline with slope $m = -4$.

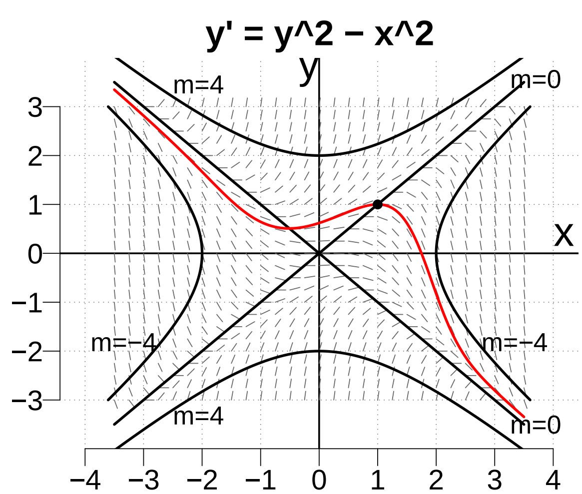


(a) *Sketch the nullcline (isocline with slope $m = 0$).* Clearly label your answer.

Solution: See picture below.

(b) *Sketch in the solution curve with $y(1) = 1$.*

Solution: See picture at below.



(c) *Suppose you used Euler's method to estimate $y(1.2)$ for your solution in Part (b). Is* *the estimate too high or too low?* Give a reason for your answer.

Solution: We'll use the second derivative to understand the convexity of y.

$y' = y^2 - x^2 \quad \Rightarrow y'' = 2y\,y' - 2x.$

Thus, $y(1) = 1 \quad \Rightarrow y'(1) = 0 \quad \Rightarrow y''(1) = -2$

$y''(1) < 0 \Rightarrow$ concave down (also see plot above).

Concave down, so approximation is probably an overestimate.

(d) *Estimate $y(100)$, where y is the solution in Part (b).*

Give a reason for your answer.

Solution: $y(100) \approx -100$, because the integral curve is squeezed between the isoclines with $m = 0$ and $m = -4$, since on both isoclines the 'hairs' point towards the curve.

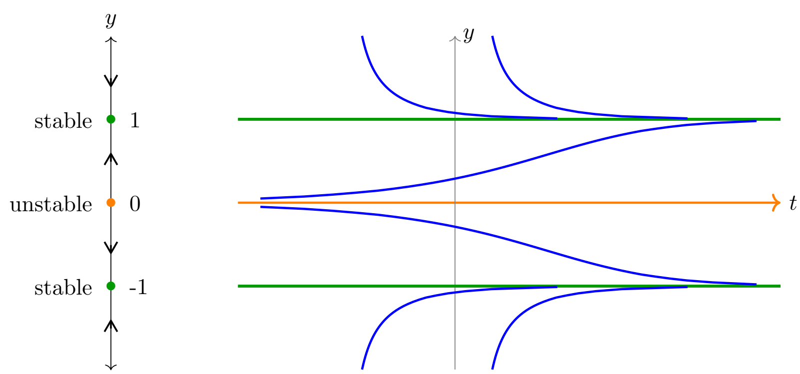
Problem 9.
The DE in this problem is $y' = ay - y^3$.

(a) *First take $a = 1$, and find and classify the critical points, give a phase line diagram and a sketch of some representative solutions.*

Solution: Critical points: $y' = y - y^3 = y(1 - y^2) = 0 \quad \Rightarrow y = 0,\, 1,\, -1.$

To draw a phase line, we check the sign of y':

$$y' = y(1-y^2) \text{ is } \begin{cases} \text{negative} & \text{for } x > 1 \\ \text{positive} & \text{for } 0 < x < 1 \\ \text{negative} & \text{for } -1 < x < 0 \\ \text{positive} & \text{for } x < -1 \end{cases}$$



(b) *Now letting the parameter a vary: draw the bifurcation diagram. Be sure to include the followin*

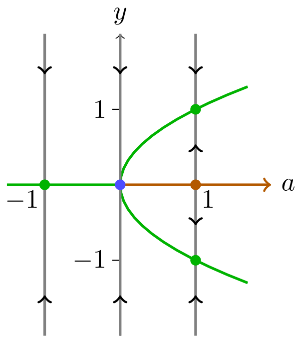
(i) Label the axes.

(ii) On the diagram add phase lines at $a = -1,\, 0,\, 1$

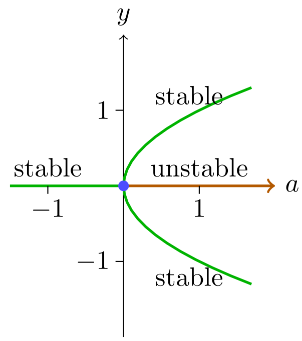
(Hint: reuse your answer to Part (a).)

Solution: The critical points are $y' = ay - y^3 = 0$. Factoring: $y(a - y^2) = 0$, so either $y = 0$ or $a = y^2$. We plot both these curves on the bifurcation diagram.

In the plot below on the left, we added phase lines for $a = -1, 0, 1$. The phase line for $a = 1$ is sufficient to tell us the sign of y' for all values of a and y. This allows us to label the stability of the critical points in the right-hand diagram below.



Bifurcation diagram with phase lines



Bifurcation diagram

(c) *(i) What does the plot represent?*

(ii) If this is a population model for what values of a is the population sustainable?

Solution: (i) The bifurcation diagram shows the critical points for each choice of the parameter a. It includes information about the stability of the critical points.

(ii) For $a > 0$, there are positive stable equilibra. So it is sustainable with these values of a.

For $a \leq 0$, there are no positive stable equilibra. So the population is unsustainable for these values of a.

Problem 10.
Solve $\ddot{x} - x = \delta(t-2) + u(t-4)$ with rest IC using. ($u(t)$ is the unit step function.)

Solution: We use superposition to solve the problem in two pieces. Note: we make use of the fact that the superposition of two solutions each satisfying rest IC also satisfies the rest IC.

The homogeneous equation $\ddot{x} - x = 0$ has solution $c_1 e^t + c_2 e^{-t}$.

First, solve $\ddot{x}_1 - x_1 = \delta(t-2)$ with rest IC. The input $\delta(t-2)$ causes a jump in $\dot{x}_1$ at $t = 2$. So, we break the solution into cases.

For $t < 2$: DE: $\ddot{x}_1 - x_1 = 0$, IC: $x_1(0) = 0$, $\dot{x}_1(0) = 0$.

This has solution $x_1(t) = 0$, for $t < 0$.

For the next case, we note $x_1(2^-) = 0$, $\dot{x}_1(2^-) = 0$.

For $t > 2$: the DE and post-initial conditions are

$$\ddot{x}_1 - x_1 = 0; \qquad x_1(2^+) = 0,\ \dot{x}_1(2^+) = \dot{x}_1(2^-) + 1.$$

Because of time invariance, we can write the homogeneous solution as $x_1(t) = c_1 e^{t-2} + c_2 e^{-(t-2)}$. (The shift by 2 is not necessary it just makes the computation a little easier.) Using the post-IC at 2^+ we get $c_1 = 1/2$, $c_2 = -1/2$.

The full solution for x_1 is

$$x_1(t) = \begin{cases} 0 & \text{for } t < 2 \\ \frac{1}{2}e^{t-2} - \frac{1}{2}e^{-(t-2)} & \text{for } t > 2. \end{cases}$$

Second, solve $\ddot{x}_2 - x_2 = u(t-4)$ with rest IC.

<u>For $t < 4$</u>: DE: $\ddot{x}_2 - x_2 = 0$, IC: $x_2(0) = 0$, $\dot{x}_2(0) = 0$

This has solution $x_2(t) = 0$ for $t < 4$.

<u>For $t > 4$</u>: The DE and IC are

$$\ddot{x}_2 - x_2 = 1; \qquad x_2(4) = \dot{x}_2(4) = 0.$$

We easily find the solution $x_2 = -1 + c_1 e^{t-4} + c_2 e^{-(t-4)}$. And, using the IC at $t = 4$, we get $c_1 = c_2 = 1/2$. So,

$$x_2(t) = \begin{cases} 0 & \text{for } t < 4 \\ -1 + \frac{1}{2}e^{t-4} + \frac{1}{2}e^{-(t-4)} & \text{for } t > 4. \end{cases}$$

Full solution to problem:

$$x(t) = x_1 + x_2 = \begin{cases} 0 & \text{for } t < 2 \\ \frac{1}{2}e^{t-2} - \frac{1}{2}e^{-(t-2)} & \text{for } 2 < t < 4 \\ \frac{1}{2}e^{t-2} - \frac{1}{2}e^{-(t-2)} - 1 + \frac{1}{2}e^{t-4} + \frac{1}{2}e^{-(t-4)} & \text{for } 4 < t \end{cases}$$

Problem 11.

(a) *The function $f(t)$ is periodic with period 2. On the interval $-1 \le t < 1$ we have $f(t) = t$. Find the Fourier series for $f(t)$.*

Solution: We have $L = 1$ and $f(t)$ is odd $\Rightarrow$ $f(t) = \sum b_n \sin(n\pi t)$, where

$$\begin{aligned} b_n &= 2\int_0^1 f(t)\sin(n\pi t)\,dt \\ &= 2\int_0^1 t\sin(n\pi t)\,dt = 2\left[-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{(n\pi t)^2}\right]_0^1 \\ &= (-1)^{(n+1)}\frac{2}{n\pi}. \end{aligned}$$

$$\Rightarrow \boxed{f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{\sin(n\pi t)}{n}.}$$

(b) *Find a periodic solution to $x'' + 36x = f(t)$.*

Solution: We will use superposition, so first we solve individual equations:

$$x_n'' + 36x_n = \sin(n\pi t).$$

We have $P(in\pi) = 36 - n^2\pi^2$. So, $|P(in\pi)| = |36 - n^2\pi^2|$, $\boxed{\phi(n) = \mathrm{Arg}(P(in\pi)) = \begin{cases} 0 & \text{if } n = 1 \\ \pi & \text{if } n > 1 \end{cases}.}$

Thus, by the SRF, $x_{n,p}(t) = \dfrac{\sin(n\pi t - \phi(n))}{|36 - n^2\pi^2|}$.

Now by superposition: $\boxed{x_p(t) = \dfrac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{(n+1)}\dfrac{\sin(n\pi t - \phi(n))}{n|36 - n^2\pi^2|}.}$

(c) *Which frequency in the Fourier series for $f(t)$ is closest to resonance for the system in Part (b).*

Solution: The natural frequency is $\sqrt{36} = 6$. The frequency of the n^{th} term is $n\pi \Rightarrow$ the term with $\boxed{n = 2}$ is closest to resonance

Problem 12.

Match each of the following Fourier series with a graph below. For credit you must give a short explanation of your choice.

(a) $4\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nt) + 4\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(nt)$

Solution: Neither even nor odd function $\Rightarrow$ $\boxed{\text{Graph II.}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\pi n^2}\cos(n\pi t)$

Solution: Even function, period 2 $\Rightarrow$ $\boxed{\text{Graph V.}}$

(c) $\displaystyle\frac{4}{\pi}\sum_{n\,odd} \frac{\sin(nt)}{n}.$

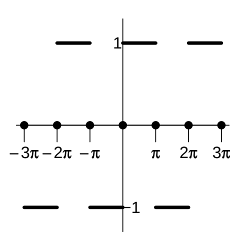
Solution: Odd period 2π square wave has known Fourier series $\Rightarrow$ $\boxed{\text{Graph I.}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\pi n^2}\cos(nt)$

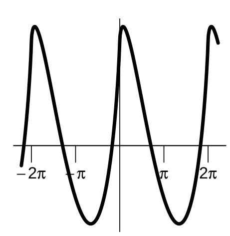
Solution: Even function, period 2π $\Rightarrow$ $\boxed{\text{Graph IV.}}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\pi n^3}\sin(nt)$

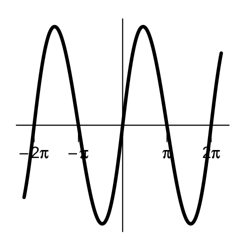
Solution: Odd function, not the square wave $\Rightarrow$ $\boxed{\text{Graph III.}}$



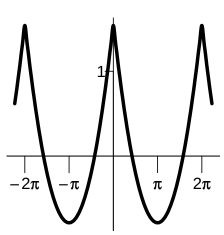
Graph I



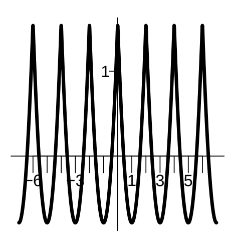
Graph II



Graph III



Graph IV



Graph V

Problem 13.
Consider the following partial differential equation with boundary and initial conditions:

PDE: $u_t(x,t) + u(x,t) = u_{xx}(x,t)$; *defined for* $0 < x < 1$.

BC: $u(0,t) = 0, \quad u(1,t) = 0.$

IC: $u(x,0) = f(x).$

(a) *The separation of variables technique looks for solutions to the PDE of the form* $u(x,t) = X(x)T(t)$. *Give the ordinary DEs satisfied by* X *and* T.

You do not have to solve these DEs.

Solution: Separated solution: $u(x,t) = X(x)T(t)$.

Substitution: $XT' + XT = X''T \Rightarrow \dfrac{T' + T}{T} = \dfrac{X''}{X} = -\lambda$ for some constant λ.

(We must have a constant because we have a function of t = a function of x.)

$\Rightarrow \boxed{X'' + \lambda X = 0 \quad \text{and} \quad T' + (1+\lambda)T = 0.}$

(b) *To make your life easier, we'll tell you that, in the usual notation, the only separated solutions satisfying the boundary conditions have* $\lambda > 0$ *and are of the form*

$$X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \ \ \textit{and}\ \ T(t) = e^{-(1+\lambda)t}.$$

Of course, not all $\lambda > 0$ *work. Find all the separated solutions to the PDE that satisfy the boundary conditions. Then give the general solution to the PDE with BC.*

Solution: For separated solutions the BC are $X(0) = 0,\ X(1) = 0.$

BC: $X(0) = a = 0, \quad X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0.$

With $a = 0$, the second equation becomes $b\sin(\sqrt{\lambda}) = 0$. If $b = 0$, we have a trivial solution. So, the nontrivial solutions have $\sin(\sqrt{\lambda}) = 0, \Rightarrow \sqrt{\lambda} = n\pi$ for some positive integer n.

Modal solutions: $\boxed{u_n(x,t) = b_n \sin(n\pi x)\, e^{-(1+n^2\pi^2)t} \ \text{ for } \ n = 1,\ 2,\ 3,\ \ldots}$.

General solution by superposition: $\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-(1+n^2\pi^2)t}}$.

(c) *Give the Fourier solution to PDE with BC and IC. Be sure to write down the integral formula for any coefficients used. (Since* f *is not specified you cannot compute the integrals.)*

Solution: IC: $u(x,0) = \sum b_n \sin(n\pi x) = f(x)$. So, b_n = Fourier sine coefficient of f:

$$\boxed{b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx.}$$

(d) *We can add input to the PDE:* $u_{xx} = u_t + u + xe^{-t}$.

A particular solution to this PDE also satisfying the BC of Part (a) is: $u_p(x,t) = \left(\dfrac{x^3}{6} - \dfrac{x}{6}\right)e^{-t}$.

What are all the solutions to this PDE which also satisfy the BC?

Solution: Using linearity, the general solution is the particular solution + the general homogeneous solution found in Part (c): $\boxed{u(x,t) = \left(\frac{x^3}{6} - \frac{x}{6}\right) e^{-t} + \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-(1+n^2\pi^2)t}.}$

Note well: When using superposition, you need to make sure that it applies to both the PDE and the BC.

Problem 14.
Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$. We know the eigenvalues and eigenvectors of A are

$$\lambda_1 = 1, \ \mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = 7, \ \mathbf{v_2} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Solve the initial value problem: $\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \mathbf{x}; \ \mathbf{x}(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Solution: Using the eigenstuff, the general solution for the system is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

The initial conditions give, $\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Solving this using your favorite method, you should find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -32 \\ 10 \end{bmatrix}$. Therefore, $\boxed{\mathbf{x}(t) = -\frac{32}{7} e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{10}{7} e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.}$

Problem 15.
For the DE system $\quad x' = -x - y + xy, \qquad y' = 2x - xy$

(a) *Show that (0,0) and (2, 2) are its only critical points.*

Solution: Critical points must satisfy $x' = -x - y + xy = 0$ and $y' = 2x - xy = 0$.

Factoring the second equation: $y' = x(2 - y) = 0$. So $x = 0$ or $y = 2$.

Using this in the first equation: $x = 0 \Rightarrow y = 0$ and $y = 2 \Rightarrow x = 2$.

Thus $(0,0)$ and $(2,2)$ are the only critical points.

(b) *Compute the linearized system at each of the critical points and solve for the eigenvalues. Solve for the eigenvectors only if they will be needed in order to get a good sketch of the trajectories in Part (d).*

Solution: Jacobian $J(x,y) = \begin{bmatrix} -1 + y & -1 + x \\ 2 - y & -x \end{bmatrix}$.

At $(0,0)$: $\quad J(0,0) = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + \lambda + 2 = 0 \quad \Rightarrow \lambda = \frac{-1 \pm \sqrt{1-8}}{2}$.

This is a linearized spiral sink (counterclockwise).

At $(2,2)$: $\quad J(2,2) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$.

Triangular, so eigenvalues are the diagonal entries: $\quad \lambda = 1, -2$.

This is a linearized saddle. We'll need eigenstuff in Part (d).

Basic eigenvectors.

$\lambda = 1$: $\quad A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$. Basic eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

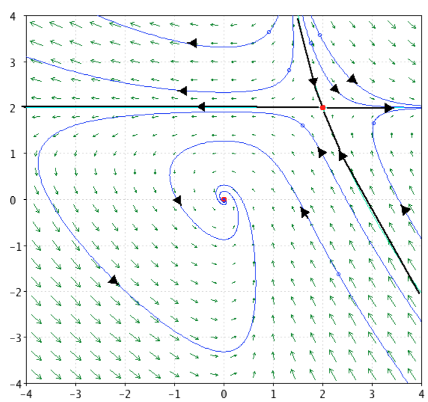
$\lambda = -2$: $\quad A - \lambda I = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$. Basic eigenvector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

(c) *Will the behavior of the trajectories of non-linear system near the critical points be essentially the same as that of the linearized system in each case? What property of the linearized system at the critical point allows you to be able to tell in each case?*

Solution: Yes, both linearized critical points are structurally stable.

(d) *Using all the information about the linearized system at the critical points found in Parts (b) and (c), sketch in (on the x-y plot below) some trajectories in the neighborhood of each critical point. Then use this to create a conjectural phase portrait of the non-linear system.*

Solution: See figure.



Problem 16.

A model for the spread of a disease, which travels between two species S_1 and S_2, is given by

$$x' = -2x + a(1-x)y, \quad y' = -y + a(1-y)x \text{ with } a > 0.$$

Here $x(t)$ represents the fraction of the S_1 population which is carrying the disease and $y(t)$ is the corresponding fraction of the population S_2. The expressions $(1-x)y$ and $(1-y)x$ measure encounters between the infected and uninfected portions of the two populations, and the parameter a measures the transmission rate. Note that the disease would die out exponentially in each population were it not for infection from the other (i.e., if $a = 0$ it dies out).

(a) *For $a = 1$, the only critical point with physical significance for this model is (0,0). Find the type of this critical point in the linearized approximation to this system.*

Solution: For $a = 1$ we have $x' = -2x + y - xy$ and $y' = -y + x - xy$.

$$J(x,y) = \begin{bmatrix} -2-y & 1-x \\ 1-y & -1-x \end{bmatrix} \Rightarrow J(0,0) = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Characteristic equation: $\lambda^2 + 3\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{-3 \pm \sqrt{5}}{2} \Rightarrow$ $\boxed{\text{linearized nodal sink.}}$

(b) *For $a = 2$, (0,0) and $\left(\frac{1}{4}, \frac{1}{3}\right)$ are the only critical points. Again, find their types in the linearized approximation to this system.*

Solution: For $a = 2$ we have $x' = -2x + 2y - 2xy$ and $y' = -y + 2x - 2xy$.

$$J(x,y) = \begin{bmatrix} -2-2y & 2-2x \\ 2-2y & -1-2x \end{bmatrix}.$$

At $(0,0)$: $J(0,0) = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 3\lambda - 2 = 0$. Because the determinant is negative, we know this is a $\boxed{\text{linearized saddle}}$.

At $(1/4, 1/3)$: $J(1/4, 1/3) = \begin{bmatrix} -8/3 & 3/2 \\ 4/3 & -3/2 \end{bmatrix}$

Characteristic equation: $\lambda^2 + \dfrac{25}{6}\lambda + 2 = 0$. Roots are real and negative, so this is a $\boxed{\text{linearized nodal sink.}}$

(c) *What long-range outcome does this analysis predict for the long-term levels of the disease in the populations, for the transmission rates $a = 1$ and $a = 2$. What is the effect of the increased communicablility of the disease?*

Solution: Since all the linearized critical points are structurally stable, the critical points in the nonlinear system are each of the same type as their linearized versions.

For $a = 1$: the lone critical point at the origin is asymptotically stable, so we expect all trajectories to go to $(0,0)$, i.e., in the long-term the disease disappears.

For $a = 2$: the only stable critical point is the sink at $(1/4, 1/3)$, so we expect that in the long-term the disease will be endemic with $1/4$ of S_1 and $1/3$ of S_2 sick at any one time.

The increased communicablility allows the disease to stay in the population instead of dying out.

End practice final 2 solutions