# ES.1803 Practice 2 Final Quiz, Spring 2024

This practice contains no linear algebra problems.

On the final quiz, you will be given the formula posted alongside these problems.

**Problem 1.**

**(a)** Find *all* solutions to the DE $\dfrac{dy}{dx} = y^2$.

**(b)** Give a definite integral solution to the following IVP

$$y' + 4\,y = \cos(t^2 + t^3);\ \ y(0) = 2.$$

*You do not have to evaluate the integral.*

**Problem 2.**

**(a)** Give the general *real* solution to $\ddot{x} + 3\dot{x} + 4x = e^{2t} + t + 3$.

**(b)** If the differential operator in Part (a) models a physical system is the system stable? Eplain how you know.

**(c)** What is the amplitude response for the system $\ddot{x} + 3\dot{x} + 4x = \cos(\omega t)$, where $\cos(\omega t)$ is the input?

**(d)** For what values of $k$ does $\ddot{x} + 3\dot{x} + kx = 0$ have oscillatory solutions?

**Problem 3.** Let $P(D)$ be a constant coefficient differential operator.

Suppose that the DE $P(D)x = f$ with $f(t) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(nt)$ has the periodic solution

$$x_p(t) = \sum_{n=1}^{\infty} \frac{1}{|5-n^2|(n^2)}\sin(nt - \phi(n)), \quad \text{where } \phi(n) = \begin{cases} 0 & \text{for } n < \sqrt{5} \\ \pi & \text{for } n > \sqrt{5} \end{cases}.$$

**(a)** Without finding $P(D)$ write down the periodic solution to the DE $P(D)x = \sin(3t)$.

**(b)** Find $P(D)$.

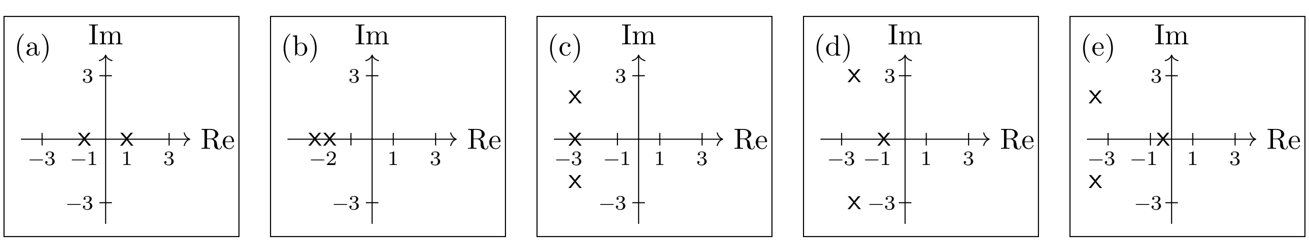
**Problem 4.** Solve $x'' + 5x = 5\cos(\omega t)$. *(Be sure to do this for every value of $\omega$.)*

**Problem 5.**

Solve $(D^3 + I)y = 0$.

**Problem 6.**

Consider the following pole diagrams for 5 linear time invariant systems of the form $P(D)y = f$. (The diagrams are in the complex $s$-plane, where we consider the characteristic polynomial $P(s)$ to be a function of $s$.)



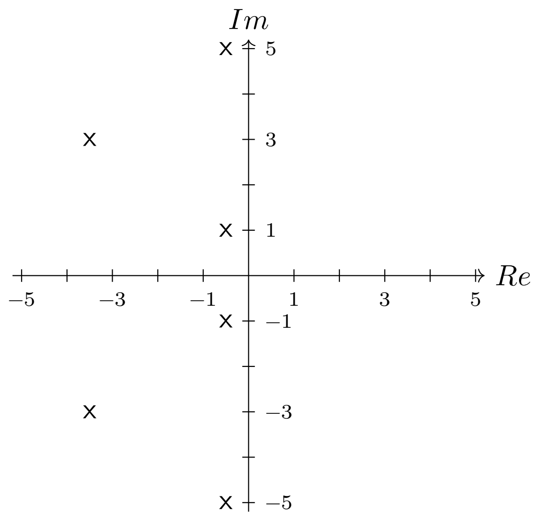
**(a)** List all the stable systems.

**(b)** Choose the stable system where the transient has the fastest decay.

**(c)** Choose the stable system where the transient decays as fast as possible without oscillation.

**(d)** Below is the pole diagram for a linear time invariant system with cosine input: $P(D)x = F_0\cos(\omega t)$. In this particular system $\omega$ must be an integer between 1 and 5 and it is critical that the response be kept as *small* as possible.

What frequency $\omega$ would you use? Mark the pole diagram with a solid dot in the appropriate place to indicate this frequency.



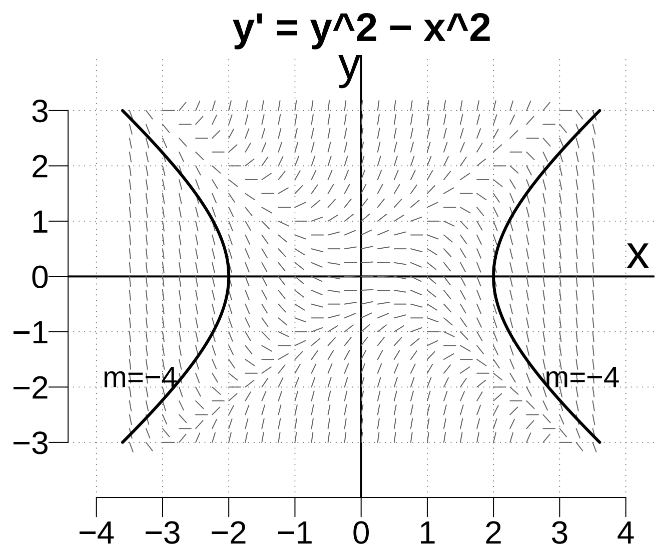

**Problem 7.**
For a linear, time invariant system, if the input is $e^{st}$, then the output is $G(s)e^{st}$. The gain factor $G(s)$ is called the complex gain of the system.

Suppose such a system has complex gain $G(s) = \dfrac{s}{(s^2+9)(s+7)(s+1)}$.

**(a)** For what value of $\omega > 0$ will the input $\cos(\omega t)$ give the biggest response.

**(b)** The system with input $f$ satisfies the DE $P(D)y = f'$. What is $P(D)$?

**Problem 8.**
The DE for this problem is $y' = y^2 - x^2$. The direction field for this DE is shown. We also show the isocline with slope $m = -4$.




**(a)** Sketch the nullcline (isocline with slope $m = 0$). *Clearly label your answer.*

**(b)** Sketch in the solution curve with $y(1) = 1$.

**(c)** Suppose you used Euler's method to estimate $y(1.2)$ for your solution in Part (b). Is the estimate too high or too low? *Give a reason for your answer.*

**(d)** Estimate $y(100)$, where $y$ is the solution in Part (b).

*Give a reason for your answer.*

**Problem 9.**
The DE in this problem is $y' = ay - y^3$.

**(a)** First take $a = 1$, and find and classify the critical points, give a phase line diagram and a sketch of some representative solutions.

**(b)** Now letting the parameter $a$ vary: draw the bifurcation diagram. Be sure to include the followin

(i) Label the axes.

(ii) On the diagram add phase lines at $a = -1, 0, 1$

(Hint: reuse your answer to Part (a).)

**(c)** (i) What does the plot represent?

(ii) If this is a population model for what values of $a$ is the population sustainable?

**Problem 10.**
Solve $\ddot{x} - x = \delta(t-2) + u(t-4)$ with rest IC using. ($u(t)$ is the unit step function.)

**Problem 11.**
**(a)** The function $f(t)$ is periodic with period 2. On the interval $-1 \le t < 1$ we have $f(t) = t$. Find the Fourier series for $f(t)$.

**(b)** Find a periodic solution to $x'' + 36x = f(t)$.

**(c)** Which frequency in the Fourier series for $f(t)$ is closest to resonance for the system in Part (b).

**Problem 12.**
Match each of the following Fourier series with a graph below. For credit you must give a *short* explanation of your choice.

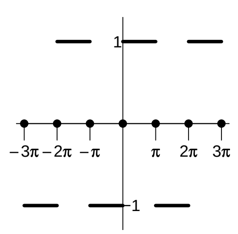
**(a)** $4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nt)+4\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nt)$

**(b)** $\sum_{n=1}^{\infty}\frac{3}{\pi\, n^2}\cos(n\pi t)$

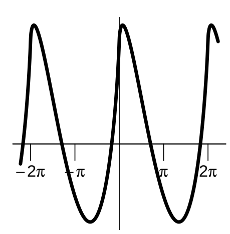
**(c)** $\frac{4}{\pi}\sum_{n\,odd}\frac{\sin(nt)}{n}$.

**(d)** $\sum_{n=1}^{\infty}\frac{3}{\pi\, n^2}\cos(nt)$

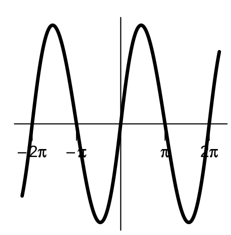
**(e)** $\sum_{n=1}^{\infty}\frac{3}{\pi\, n^3}\sin(nt)$



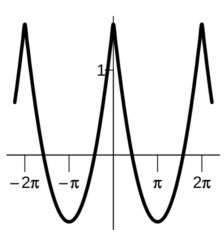
Graph I



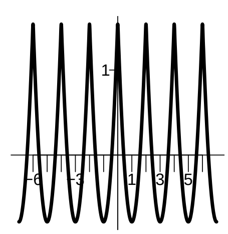
Graph II



Graph III



Graph IV



Graph V

**Problem 13.**
Consider the following partial differential equation with boundary and initial conditions:

PDE: $u_t(x,t)+u(x,t)=u_{xx}(x,t)$; defined for $0<x<1$.

BC: $u(0,t)=0$, $u(1,t)=0$.

IC: $u(x,0)=f(x)$.

**(a)** The separation of variables technique looks for solutions to the PDE of the form $u(x,t)=X(x)T(t)$. Give the ordinary DEs satisfied by $X$ and $T$.

*You do not have to solve these DEs.*

**(b)** To make your life easier, we'll tell you that, in the usual notation, the only separated solutions satisfying the boundary conditions have $\lambda > 0$ and are of the form

$$X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \text{ and } T(t) = e^{-(1+\lambda)t}.$$

Of course, not all $\lambda > 0$ work. Find all the separated solutions to the PDE that satisfy the boundary conditions. Then give the general solution to the PDE with BC.

**(c)** Give the Fourier solution to PDE with BC and IC. Be sure to write down the integral formula for any coefficients used. (Since $f$ is not specified you cannot compute the integrals.)

**(d)** We can add input to the PDE: $u_{xx} = u_t + u + xe^{-t}$.

A particular solution to this PDE also satisfying the BC of Part (a) is: $u_p(x,t) = \left(\frac{x^3}{6} - \frac{x}{6}\right)e^{-t}$.

What are all the solutions to this PDE which also satisfy the BC?

**Problem 14.**
Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$. We know the eigenvalues and eigenvectors of $A$ are

$$\lambda_1 = 1,\ \mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 7,\ \mathbf{v_2} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Solve the initial value problem: $\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\mathbf{x}$; $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

**Problem 15.**
For the DE system $\quad x' = -x - y + xy, \qquad y' = 2x - xy$

**(a)** Show that (0,0) and (2, 2) are its only critical points.

**(b)** Compute the linearized system at each of the critical points and solve for the eigenvalues. Solve for the eigenvectors only if they will be needed in order to get a good sketch of the trajectories in Part (d).

**(c)** Will the behavior of the trajectories of non-linear system near the critical points be essentially the same as that of the linearized system in each case? What property of the linearized system at the critical point allows you to be able to tell in each case?

**(d)** Using all the information about the linearized system at the critical points found in Parts (b) and (c), sketch in (on the x-y plot below) some trajectories in the neighborhood of each critical point. Then use this to create a conjectural phase portrait of the non-linear system.

**Problem 16.**
A model for the spread of a disease, which travels between two species $S_1$ and $S_2$, is given by

$$x' = -2x + a(1-x)y, \quad y' = -y + a(1-y)x \text{ with } a > 0.$$

Here $x(t)$ represents the fraction of the $S_1$ population which is carrying the disease and $y(t)$ is the corresponding fraction of the population $S_2$. The expressions $(1-x)y$ and $(1-y)x$ measure encounters between the infected and uninfected portions of the two populations, and the parameter $a$ measures the transmission rate. Note that the disease would die out exponentially in each population were it not for infection from the other (i.e., if $a = 0$ it dies out).

**(a)** For $a = 1$, the only critical point with physical significance for this model is (0,0). Find the type of this critical point in the linearized approximation to this system.

**(b)** For $a = 2$, (0,0) and $\left(\frac{1}{4}, \frac{1}{3}\right)$ are the only critical points. Again, find their types in the linearized approximation to this system.

**(c)** What long-range outcome does this analysis predict for the long-term levels of the disease in the populations, for the transmission rates $a = 1$ and $a = 2$. What is the effect of the increased communicablility of the disease?