ES.1803 Practice Solutions – Quiz 2, Spring 2024

Problem 1.

(a) *Compute the following real function of x:* $\text{Im}\left(\dfrac{e^{(3+2i)x}}{3+2i}\right)$.

(As usual, Im(z) denotes the imaginary part of the complex number z.)

Solution: In polar coordinates: $3+2i = \sqrt{13}e^{i\phi}$, where $\boxed{\phi = \text{Arg}(2+3i) = \tan^{-1}(2/3) \text{ in 1st quadrant.}}$

So our function equals

$$\text{Im}\left(\frac{e^{3x}}{\sqrt{13}}e^{(2x-\phi)i}\right) = \boxed{\frac{e^{3x}}{\sqrt{13}}\sin(2x-\phi).}$$

(In rectangular coordinates: this equals $\dfrac{e^{3x}}{13}(3\sin(2x) - 2\cos(2x))$.)

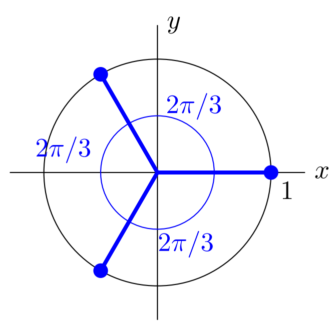
(b) *Use the result of Part (a) to compute the integral* $\int e^{3x}\sin(2x)\,dx$ *using the complex exponential.*

Solution: $\int e^{3x}\sin(2x)dx = \text{Im}\left(\int e^{(3+2i)x}dx\right) = \text{Im}\left(\dfrac{e^{(3+2i)}}{3+2i}\right)$. This last term is the same as the expression in Part (a): $\boxed{\dfrac{e^{3x}}{\sqrt{13}}\sin(2x-\phi) = \dfrac{e^{3x}}{13}(3\sin(2x) - 2\cos(2x)).}$

Problem 2. *Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.*

Solution: The picture shows the roots evenly spaced around the unit circle. They are

$$1,\ e^{i2\pi/3},\ e^{i4\pi/3} \quad = \quad 1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$



Problem 3. *Use Euler's formula to derive the trig addition formulas for* sin *and* cos.

Solution: We look at $e^{ia}e^{ib}$ in two different ways.

$$\begin{aligned} e^{ia}e^{ib} &= (\cos(a) + i\sin(a))(\cos(b) + i\sin(b)) \\ &= \cos(a)\cos(b) - \sin(a)\sin(b) + i\left[\cos(a)\sin(b) + \cos(b)\sin(b)\right] \\ e^{ia}e^{ib} &= e^{i(a+b)} = \cos(a+b) + i\sin(a+b) \end{aligned}$$

Comparing the right-hand side of both equations gives the trig addition formulas.

Problem 4.
(a) *Find the general* real-valued *solution to the DE* $y'' + 4y' + 13y = 0$. *Also find the solution satisfying the intial conditions (IC)* $y(0) = 1,\ y'(0) = 0$.

Solution: Characteristic equation: $P(r) = r^2 + 4r + 13 = 0$, so the roots are

$$r = (-4 \pm \sqrt{-36})/2 = -2 \pm 3i.$$

Two real-valued solutions: $y_1(t) = e^{-2t}\cos(3t),\ y_2(t) = e^{-2t}\sin(3t)$.

General real-valued solution: $\boxed{y(t) = c_1y_1 + c_2y_2 = c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t).}$

Using some algebra to find the coeffients c_1 and c_2, we find that the solution satisfying the IC is

$$\boxed{y(t) = e^{-2t}\cos(3t) + \frac{2}{3}e^{-2t}\sin(3t).}$$

(b) *For what values of* b *will all the (non-zero) solutions to* $y'' + by' + 13y = 0$ *display oscillatory behavior?*

Solution: Characteristic equation: $P(r) = r^2 + br + 13 = 0$, so the roots are $r = (-b \pm \sqrt{b^2 - 52})/2$.

If the roots have an imaginary part then the (real) solutions will have sin and cos terms, i.e., oscillatory behavior. This happens if $b^2 < 52$ or $|b| < 2\sqrt{13} \approx 7.211$.

(c) *For these oscillatory solutions, in theory how many times does each solution cross the positive* t*-axis. If this DE is modeling some real-world situation, what actually happens to the quantity* $y = y(t)$ *in the long run?*

Solution: The solutions cross the t-axis infinitely often. If $b > 0$ then, since the solutions all have a factor of e^{-bt}, the amplitude of the oscillations goes to 0. In practice the system will come to a halt after some (possibly long) finite time.

Problem 5.
(a) *Give the general* real-valued *solution to the DE* $x'' + 7x' + 12x = 2e^t$.

Solution: Homogeneous DE: $x'' + 7x' + 12x = 0$.

Chararteristic equation: $P(r) = r^2 + 7r + 12 = 0$. This factors as $(r + 3)(r + 4) = 0$.

Characteristic roots: $r = -3, -4$.

Homogeneous solution: $x_h(t) = c_1e^{-3t} + c_2e^{-4t}$.

Particular solution using the exponential response formula: $x_p(t) = \dfrac{2e^t}{P(1)} = \dfrac{2e^t}{20}$.

General solution: $\boxed{x(t) = x_p + x_h = \dfrac{2e^t}{20} + c_1e^{-3t} + c_2e^{-4t}.}$

(b) (5) *Give the general* real-valued *solution to the DE* $x'' + 7x' + 12x = 2e^t + 3e^{-t}$.

Solution: Use the answer in Part (a), the exponential response formula and superposition.

General solution: $\boxed{x = \dfrac{2e^t}{20} + \dfrac{3e^{-t}}{6} + c_1 e^{-3t} + c_2 e^{-4t}.}$

Problem 6. *Consider the IVP $x'' + bx' + 4x = 0$; $x(0) = 1$; $x'(0) = 0$.*

Sketch a graph of the solution in the following cases. (You don't need to solve the DE completely or give a detailed sketch. You do need to give a small amount of explanation.)

(a) (5) *When $b = 1$*

Solution: Damped harmonic oscillator:

Characteristic equation: $r^2 + br + 4 = 0$.

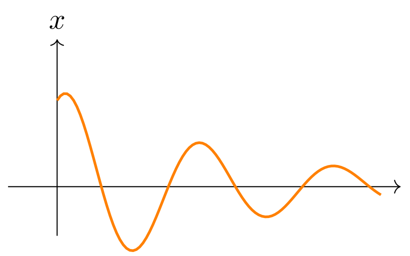
Characteristic roots: $r = (-b^2 \pm \sqrt{b^2 - 16})/2$.

$b = 1 \Rightarrow$ complex roots $\Rightarrow$ underdamped.

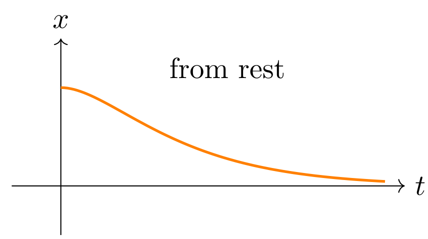
See the figure with Part (b)

(b) (5) *When $b = 6$.*

Solution: $b = 6 \Rightarrow$ real roots $\Rightarrow$ overdamped.



(a) Underdamped spring system. Plot crosses x-axis infinitely often.



(b) Overdamped spring system.

Problem 7. *Given the DE $y'' + 4y' + 5y = 8\cos(2t)$:*

(a) *Find the general solution to the DE.*

Solution: Homogeneous solution: Roots of $P(r) = r^2 + 4r + 5$ are $r = -2 \pm i$.

So, $y_h(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t))$.

Particular solution: Method 1. Use the sinusoidal response formula directly.

$$y_p(t) = \frac{8}{|P(2i)|}\cos(2t - \phi), \text{ where } \phi = \text{Arg}(P(2i)).$$

$P(2i) = 1 + 8i$, so $\boxed{|P(2i)| = \sqrt{65}}$, and $\boxed{\phi = \tan^{-1}(8) \approx 83^\circ}$.

Method 2. Use complexification and the exponential response formula to find the same formula.

Complexify: $P(D)z = 8e^{i2t}$. with $y = \text{Re}(z)$.

Exponential response formula: $P(2i) = 1 + 8i = \sqrt{65}e^{i\phi}$, where $\phi = \tan^{-1}(8) \approx 83^\circ$.

The ERF gives $z_p(t) = 8\frac{e^{i2t}}{P(2i)} = \frac{8e^{i2t}}{\sqrt{65}e^{i\phi}} = \frac{8}{\sqrt{65}}e^{i(2t-\phi)}$. Thus,

$$y_p(t) = \text{Re}(z_p(t)) = \frac{8}{\sqrt{65}}\cos(2t - \phi).$$

General solution to the DE:

$$\boxed{y(t) = y_p(t) + y_h(t) = \frac{8}{\sqrt{65}}\cos(2t-\phi) + c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t).}$$

(b) *What is the periodic solution? Give your answer in amplitude-phase form.*

Solution: Periodic solution is $\boxed{y_p(t) = \frac{8}{\sqrt{65}}\cos(2t-\phi).}$

(c) *Show that, no matter what the initial conditions, the system always settles down to the periodic solution.*

Solution: Since all the exponents in homogeneous solution have negative real parts, $y_h(t) \to 0$ as $t \to \infty$ (for any choice of c_1, c_2). That is, $y(t) \to y_p(t)$ for any solution $y(t)$.

Problem 8. *Find the general solution to $x' + 3x = t^2 + 3$*

Solution: Try $x = At^2 + Bt + C$. Substituting we get $(2At + B) + 3(At^2 + Bt + C) = t^2 + 3$. Comparing coefficients gives

$$\begin{aligned} t^2: &\quad 3A = 1 \\ t: &\quad 2A + 3B = 0 \\ 1: &\quad B + 3C = 3 \end{aligned}$$

Solving the algebraic system gives $A = 1/3$, $B = -2/9$, $C = 29/27$. So, $x_p(t) = \frac{t^2}{3} - \frac{2}{9}t + \frac{29}{27}$, The general solution (including the homogeneous piece) is

$$x(t) = x_p(t) + x_h(t) = \frac{t^2}{3} - \frac{2}{9}t + \frac{29}{27} + ce^{-3t}$$

Problem 9. *Find one solution to $x''' + 3x'' + 2x' + 5x = 4$.*

Solution: Try a constant solution: $\boxed{x(t) = 4/5.}$

Problem 10. *Let $P(D) = D^2 + bD + I$ where $D = \dfrac{d}{dt}$ and $b > 0$.*

(a) *For what range of the values of b will the solutions to $P(D)y = 0$ exhibit oscillatory behavior?*

Solution: Oscilllatory behavior means the characteristic roots are complex. The roots are $\dfrac{-b \pm \sqrt{b^2-4}}{2}$. These are complex when $b^2 - 4 < 0$. Thus, for $\boxed{0 < b < 2}$, there is oscilllatory behavior.

(b) *Describe the different types of graphs, $y = y(t)$, one gets for values of $b > 0$.*

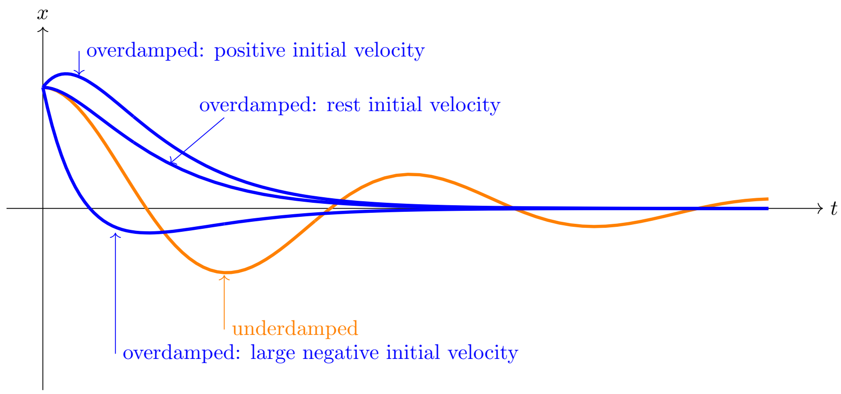
Solution: Possibilities:

Overdamped ($b > 2$): At most one turning point (derivative = 0) and at most one crossing of x-axis.

Critically damped ($b = 2$): At most one turning point (derivative $= 0$) and at most one crossing of x-axis. (At this level of precision, it is qualitatively the same as overdamped.)

Underdamped ($0 < b < 2$): Oscillatory behavior with decaying amplitude.

The following plot shows three overdamped cases depending on the initial velocity. It shows one underdamped case.



(c) *For $b = 1$, solve the DE $P(D)\,y = f(t)$ for a particular solution $y_p(t)$ where f is the following:*

(i) $f(t) = 2\,e^{-t}\sin(2t)$ (ii) $f(t) = 2\,e^{-t}\cos(2t)$ (iii) $f(t) = t + 3$.

Solution: (i) Complexify: $z'' + z' + z = 2e^{(-1+2i)t}$, $y = \mathrm{Im}(z)$.

Exponential response formula: $z_p(t) = 2\dfrac{e^{(-1+2i)t}}{P(-1+2i)}$

$P(-1+2i) = -3-2i$, so $\boxed{|P(-1+2i)| = \sqrt{13}}$ and $\boxed{\phi = \mathrm{Arg}(-3-2i) = \tan^{-1}(2/3) \text{ in Q3.}}$

Taking the imaginary part:

$$y_p(t) = \mathrm{Im}(z_p(t)) = \boxed{\frac{2e^{-t}}{|P(-1+2i)|}\sin(2t-\phi) = \frac{2}{\sqrt{13}}e^{-t}\sin(2t-\phi)}.$$

(ii) From Part (i) we have, $y_p(t) = \mathrm{Re}(z_p) = \boxed{\dfrac{2}{\sqrt{13}}e^{-t}\cos(2t-\phi)}$.

(iii) Use the method of undetermined coefficients: Try $y(t) = At + B$.

Substitute: $y' = A$, $y'' = 0$. So, $At + (A + B) = t + 3$. Equating coefficients gives $A = 1$, $B = 2$. So, $\boxed{y_p(t) = t + 2.}$

(d) *For $b = 2$, find the general solution of the DE $P(D)\,y = 0$*

Solution: Roots of $P(r)$: Double root $r = -1, -1$.

General solution: $\boxed{y(t) = c_1e^{-t} + c_2te^{-t}.}$

Problem 11. *Consider the DE $y'' + 2y' + ky = 0$.*

(a) *For which values of k will there be solutions $y(t)$ with infinitely many zeros?*

Solution: Infinitely many zeros $\Rightarrow$ oscillatory $\Rightarrow$ complex roots.
Characteristic equation: $r^2 + 2r + k = 0$.

Roots: $-1 \pm \sqrt{1-k}$. Roots are complex when $\boxed{k > 1.}$

(b) *For such solutions, express in terms of k the t-distance between successive zeros.*

Solution: For $k > 1$ let $\sqrt{k-1} = \beta$, so the roots $= 1 \pm i\beta$.

Therefore, the general solution is $y(t) = c_1 e^{-t}\cos(\beta t) + c_2 \sin(\beta t) = Ae^{-t}\cos(\beta t - \phi)$.

The period of $\cos(\beta t - \phi)$ is $2\pi/\beta$. A zero occurs every 1/2 period, so the time between zeros is $\boxed{\pi/\beta = \pi/\sqrt{k-1}}$.

(c) *For which values of k will $\lim_{t\to\infty} y(t) = 0$ for all solutions $y(t)$? (Indicate reason.)*

Solution: $y(t) \to 0$ for all solutions exactly when the real part of each root is negative.

Roots $= -1 \pm \sqrt{1-k}$

For $k < 0$ the roots are real, one positive and one negative.
For $k = 0$ the roots are -1, 0.
For $0 < k \le 1$ the roots are real and negative.
For $1 \le k$ the roots are complex with negative real part.

$\boxed{\text{Answer: all } k > 0.}$

(d) *For which values of k will $\lim_{t\to\infty} y(t) = 0$ for at least one nontrivial solution $y(t)$? Why?*

Solution: $y(t) \to 0$ for at least one non-trival solution means at least one root has negative real part. Using the answer in Part (c) we get $\boxed{\text{Answer: all values of } k.}$

Problem 12. *Assume L is a linear differential operator and y_1 is a solution to the DE $Ly = 0$. Prove that if y_p is a solution to the DE $Ly = f$, then so are all the functions $y_p + cy_1$, where c is any constant.*

Solution: Substitute $y_p + cy_1$ into the DE. The linearity of L implies:

$$L(y_p + cy_1) = Ly_p + cLy_1 = f + c \cdot 0 = f.$$

Problem 13. (*Parts (a) and (b) are not related*)

(a) *Suppose that the functions $y_1 = t$ and $y_2 = \dfrac{1}{t}$ both satisfy a certain inhomogeneous first-order linear DE. Write down the general solution to the DE.*

Solution: We know that the general solution to an inhomogeneous first-order linear DE is of the form $y_p(t) + cy_h(t)$, where y_p is one solution to the inhomogeneous DE and y_h is a solution to the associated homogeneous DE.

By linearity $y_h = y_1 - y_2$ satisfies the related homogeneous equation. Thus, $\boxed{y(t) = y_1 + cy_h = t + c(t - 1/t)}$ is the general solution.

(b) *Let T be the operator defined by $Tf = f^2$.*

(i) Show that the operators T and D do not commute.

(ii) Is T a linear operator? (You must give an explanation.)

Solution: (i) We apply $T\,D$ and $D\,T$ to an arbitrary ('test') function:

$$T(D(f)) = T(f') = (f')^2 \text{ and } D(T(f)) = D(f^2) = 2ff'.$$

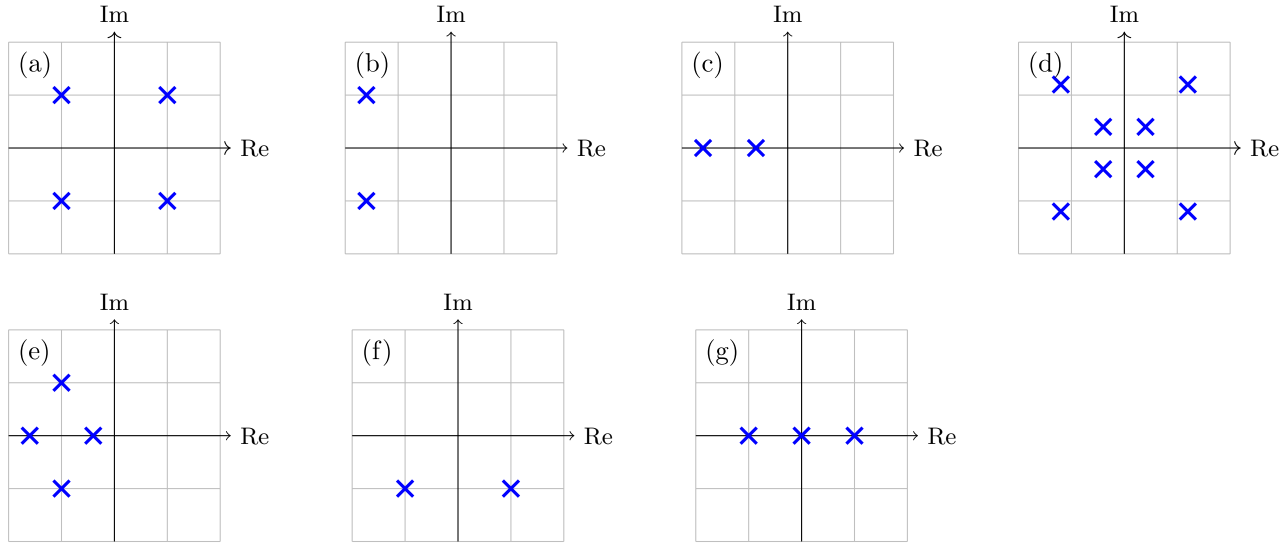
These are not the same, i.e., $T\,D \neq D\,T$, i.e., the operators don't commute.

(ii) T is not linear because $T(f+g) = f^2 + 2fg + g^2 \neq Tf + Tg$.

Problem 14.

(a) *In this problem we consider the linear constant coefficient DE $P(D)x = 0$.*

Assume $P(D)$ is of arbitrary order. Each of the following plots are in the complex plane and the crosses give the locations of the zeros of $P(r)$. If the plot comes from a stable system label it as 'stable'. If not, label it as 'unstable'.



Solution: Stable systems: b, c, e. Unstable systems: a, d, f, g.

(b) *Assume all the pole diagrams are on the same scale, which of the stable systems decays to equilibrium the fastest.*

Solution: The decay rate is determined by the right-most root. The more negative the right-most root, the greater the decay rate. Of the stable systems (b) has the largest decay rate.

Problem 15. *Water is being heated at a rate of $5^\circ C$ per minute. It is simultaneously cooling at a rate proportional to the difference between its current temperature $T(t)$ and the ambient temperature A in which it is sitting.*

(a) (5) *Write down the DE for the temperature $T(t)$. Define all letters used, with units.*

Solution: $T' = -k(T - A) + 5$. T, A in °C, k = rate constant is in 1/min.

(b) (5) *What is the long-range behavior of $T(t)$ as t increases, and why?*

Solution: We have $T' + kT = kA + 5$. So, $T(t) = A + 5/k + Ce^{-kt}$. Thus, in the long-term, T is steady at $A + 5/k$.

End of practice quiz solutions