ES.1803 Practice Solutions – Quiz 3, Spring 2024

Problem 1.
Consider the lightly damped second-order system $y'' + 2y' + 10y = \sin(\omega t)$.

For this problem, consider $\sin(\omega t)$ *to be the input to the system.*

(a) *What is the natural frequency of this system?*

Solution: The natural frequency is the frequency of the undamped, unforced harmonic oscillator $y'' + 10y = 0$. This has (angular) natural frequency $\omega_0 = \sqrt{10}$.

(b) *If the damping is removed, what* integer *frequency* ω_{int} *will make the amplitude of* $y_p(t)$ *relatively large?*

Solution: Without damping, a second order system has resonance at its natural frequency. Since $= \omega_0 = \sqrt{10} \approx 3$, $\boxed{\omega_{\text{int}} = 3}$.

Alternatively, we can compute the solution to $y'' + 10y = \sin(\omega t)$:

The SRF gives $y_p(t) = \dfrac{\sin(\omega t - \phi(w))}{|P(i\omega)|} = \dfrac{\sin(\omega t - \phi(w))}{|10 - \omega^2|}$.

The amplitude of the response is $\dfrac{1}{|10 - \omega^2|}$. This is big when the denominator is small. Therefore, the integer frequency that gives the biggest amplitude is $\boxed{\omega_{\text{int}} = 3}$.

(c) *Find any practical resonant frequencies for the damped system.*

Solution: With the damping included, the gain is

$$g(\omega) = \frac{1}{|P(i\omega)|} = \frac{1}{\sqrt{(10 - \omega^2)^2 + 4\omega^2}}.$$

Resonant frequencies are those positive ω that give relative maxima for $g(\omega)$. To find the maximum, we solve $g'(w) = 0$.

$$g(\omega) = \left((10 - \omega^2)^2 + 4\omega^2\right)^{-1/2} \quad \Rightarrow g'(\omega) = -\frac{1}{2}\left(-4\omega(10 - \omega^2) + 8\omega\right)\left((10 - \omega)^2 + 4\omega^2\right)^{-3/2}.$$

Now, $g'(\omega) = 0 \quad \Rightarrow -4(\omega(10 - \omega^2) + 8\omega = 0 \quad \Rightarrow \omega = \sqrt{8}$ or $\omega = 0$.

You can check by calculus or graphing that the $\sqrt{8}$ is a maximum (and not a minimum). (We never consider $\omega = 0$ as a resonant frequency.)

Problem 2.
Consider the DE $x'' + bx' + 5x = \cos(\omega t)$. *Assume, for this system, that* $\cos(\omega t)$ *is the input.*

(a) *For what* b *is it possible for this system to undergo pure resonance?*

Solution: Only when $b = 0$, i.e., when there is no damping.

(b) *For all values of* b *in your answer to Part (a), give the corresponding resonant frequency.*

Solution: For the undamped oscillator the resonant frequency equals the natural frequency $\omega_0 = \sqrt{5}$.

(c) *For each of the b values in the answer to Part (a), solve the DE with ω equal to the resonant frequency.*

Solution: From Part (a) the only value of b is $b = 0$. From Part (b) $\omega_0 = \sqrt{5}$ so our equation is

$$x'' + 5x = \cos(\omega_0 t).$$

(We'll write ω_0 so we won't have to write $\sqrt{5}$ repeatedly.)

Method 1: Use the SRF: We have $P(r) = r^2 + 5$, so $P(i\omega_0) = 0$. Then we have $P'(r) = 2r$, so $P'(i\omega_0) = 2i\omega_0 = 2\omega_0 e^{i\pi/2}$. The exended SRF gives (with $\phi = \text{Arg}(P'(i\omega_0)) = \pi/2$)

$$x_p(t) = \frac{t\cos(\omega_0 t - \phi)}{|P'(i\omega_0)|} = \frac{t\cos(\omega_0 t - \pi/2)}{2\omega_0} = \frac{t\sin(\omega_0 t)}{2\omega_0}.$$

Method 2: Complexify the DE and use the ERF:

$$z'' + 5z = e^{i\omega_0 t}, \quad x = \text{Re}(z).$$

As in Method 1, we need the extended version and $P'(i\omega_0) = 2i\omega_0$.

$$z_p(t) = \frac{te^{i\omega_0 t}}{P'(i\omega_0)} = \frac{te^{i\omega_0 t}}{2i\omega_0} = \frac{te^{i\omega_0 t}}{2e^{i\pi/2}\omega_0} = \frac{te^{i(\omega_0 t - \pi/2)}}{2\omega_0}.$$

Taking the real part we get $\boxed{x_p(t) = \frac{t\cos(\omega_0 t - \pi/2)}{2\omega_0} = \frac{t\sin(\omega_0 t)}{2\omega_0}}$.

Including the homogeneous solution, the general solution to the DE is

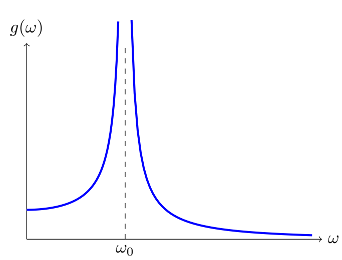
$$x(t) = x_p(t) + c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t).$$

(d) *Graph the amplitude response of each of the systems that can undergo resonance.*

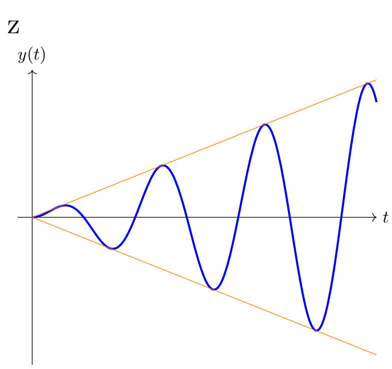
Solution: See the answer to Part (e).

(e) *Graph one nicely chosen solution from Part (c).*

Solution: We graph $x_p(t) = \frac{t\sin(\omega_0 t)}{2\omega_0}$.



(d) Undamped amplitude response



(e) Resonance response

(f) *(Unrelated to the DE above.) Assume we have a (possibly frictionless) physical system modeled by a second-order constant coefficient linear DE with positive coefficients. What criteria must the roots satisfy if this system can undergo pure resonance?*

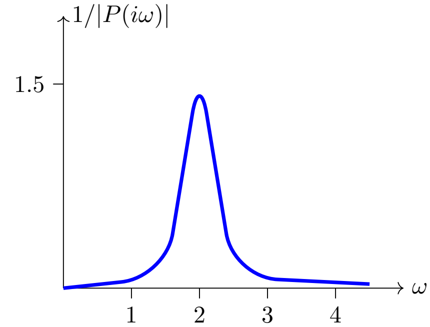
Solution: For a second-order with positive coefficients, we know that no root has a positive real part. If it can experience pure resonance, then at least one of the roots must have 0 real part.

This is noncontroversial when the root $i\beta \neq 0$. If the root is identically 0, then the response of the system to a constant input will go to infinity, just like resonance. But, perhaps we shouldn't call this resonance. An example is the velocity of a mass falling under constant gravity.

Problem 3.
Suppose that $P(r)$ is a polynomial and consider the DE

$$P(D)x = \cos t + \cos 2t + \cos 3t.$$

The graph of $\frac{1}{|P(i\omega)|}$ is shown. On the axes provided give a rough sketch of the periodic solution $x_p(t)$ to the DE. Give a brief explanation of your reasoning.



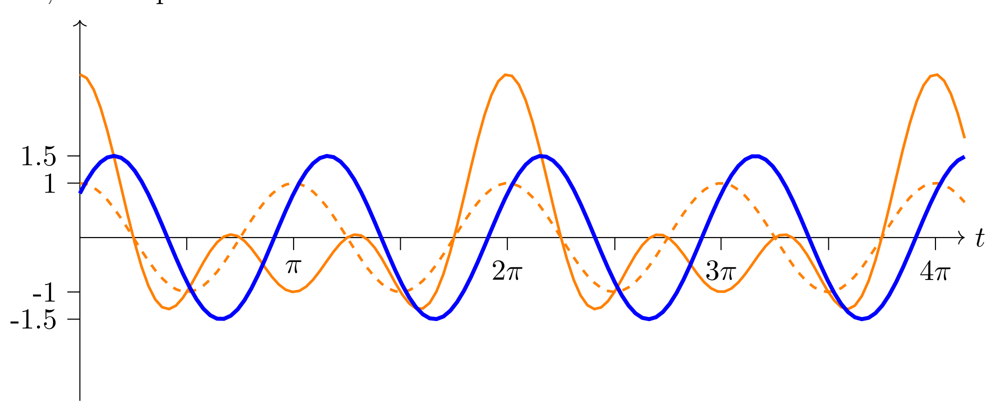
Solution: Using the sinusoidal response formula and superposition, the periodic solution to the DE is of the form $x_p(t) = \frac{\cos(t-\phi_1)}{|P(i)|} + \frac{\cos(2t-\phi_2)}{|P(2i)|}\frac{\cos(3t-\phi_3)}{|P(3i)|}$.

The graph of $1/|P(i\omega)|$ shows $1/|P(2i)| = 1.5$ and $1/|P(i)|$ and $1/|P(3i)|$ are small. Thus the system 'filters out' the frequencies 1 and 3 and the second term in x_p is by far the biggest, so

$$x_p(t) \approx \cos(2t-\phi_2)/|P(2i)| = 1.5\cos(t-\phi_2).$$

Since we don't know ϕ_2 we draw the graph with an arbitrary phase lag, but amplitude 1.5 and angular frequency 2.

The input is shown in orange, the part of the input with frequency 2 is shown as a dashed line, the output is shown in blue.



Problem 4.
The differential operator for this problem is $P(D) = D^2 + 4D + 13I$.

(a) (12) *Find the general* real-valued *solution to $P(D)x = 2\cos(\omega t)$.*

Solution: Homogeneous equation: $P(D)x = 0$.

Characteristic equation: $P(r) = r^2 + 4r + 13 = 0$.

Characteristic roots: $r = \dfrac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$.

Homogeneous solution: $\boxed{x_h(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t).}$

Particular solution using the SRF:

Preparation: $P(i\omega) = 13 - \omega^2 + 4i\omega$. So,

$|P(i\omega)| = \sqrt{(13-\omega^2)^2 + 16\omega^2}$; $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}(4\omega/(13-\omega^2)), \text{ in Q1 or Q2.}}$

So,

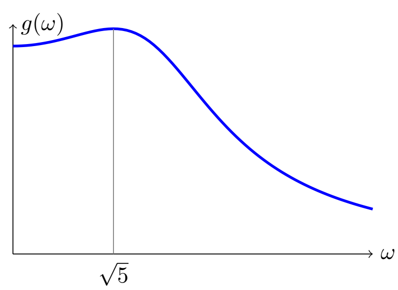
$$\boxed{x_p(t) = \frac{2\cos(\omega t - \phi(\omega))}{|P(i\omega)|} = \frac{2\cos(\omega t - \phi(\omega))}{\sqrt{(13-\omega^2)^2 + 16\omega^2}}}$$

General real-valued solution: $\boxed{x(t) = x_p(t) + x_h(t).}$

(b) (5) *Assume that $\cos(\omega t)$ is the input to the system in Part (a). Give the amplitude response (as a function of ω).*

We'll help you by saying that there is exactly one (practical) resonant frequency, which is at $\omega = \sqrt{5}$. Use this to help sketch a graph of the amplitude response. Be sure to label your axes.

Solution: From Part (a): $\boxed{g(\omega) = 2/\sqrt{(13-\omega^2)^2 + 16\omega^2}.}$ So we have $g(0) = 2/13$. The graph must increase from there to its only peak at $\omega = \sqrt{5}$. From there it must decrease asymtotically to 0 as $\omega \to \infty$.



4(b) Gain (Amplitude response)

Problem 5. *Consider the forced undamped system: $x'' + 8x = \cos(\omega t)$.*

(a) *Why is this called a forced undamped system?*

(b) *Use the exponential response formula to find a solution.*

(c) *Consider the right-hand side of the DE to be the input and graph the amplitude response function.*

(d) *What is the resonant frequency of the system?*

(e) *Why is this called the natural frequency?*

Solution: (a) No damping, input = force.

(b) (Model solution) We could use the SRF, but the instructions say to use the ERF, so we will complexify and use that.

Complexify: $z'' + 8z = e^{i\omega t}$, $x = \text{Re}\, z$.

Char. polynomial: $P(r) = r^2 + 8$. For use in the ERF and extended ERF we compute

$$P(i\omega) = 8 - \omega^2,\ |P(i\omega)| = |8 - \omega^2|,\ \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } \omega < \sqrt{8} \\ \pi & \text{if } \omega > \sqrt{8} \end{cases}$$

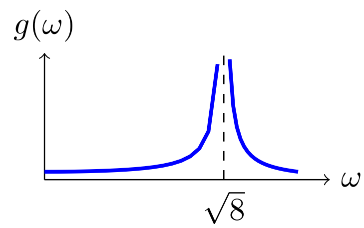
$$P'(i\omega) = 2i\omega,\ |P'(i\omega)| = 2\omega,\ \text{Arg}(P(i\omega)) = \pi/2.$$

Exponential response formula:

If $\omega \neq \sqrt{8}$, then $z_p(t) = \dfrac{e^{i\omega t}}{P(i\omega)} = \dfrac{e^{i\omega t}}{8-\omega^2} \quad \Rightarrow x(t) = \text{Re}\, z = \begin{cases} \frac{\cos(\omega t)}{|8-\omega^2|} & \text{if } \omega < \sqrt{8} \\ \frac{\cos(\omega t - \pi)}{|8-\omega^2|} & \text{if } \omega > \sqrt{8} \end{cases}$

If $\omega = \sqrt{8}$, then $z_p(t) = \dfrac{te^{i\omega t}}{P'(i\omega)} = \dfrac{te^{i\omega t}}{2i\omega} = \dfrac{te^{i\omega t}}{2\omega e^{i\pi/2}} \quad \Rightarrow x(t) = \text{Re}\, z = \dfrac{t\cos(\omega t - \pi/2)}{2\omega}$.

(c) $g(\omega) = 1/|8 - \omega^2|$. The graph of g vs. ω has a vertical asymptote at $\omega = \sqrt{8}$.



(d) The resonant frequency is at $\omega = \sqrt{8}$.

(e) Because the unforced, undamped system will naturally oscillate at this frequency

Problem 6.

Consider the forced damped second-order system: $x'' + 2x' + 9x = \cos(\omega t)$.

(a) *What is the natural frequency of the system?*

(b) *Find the response of the system in amplitude-phase form.*

(c) *Consider the right-hand side of the DE to be the input. What is the amplitude response of the system?*

(d) *What is the practical resonant frequency?*

(e) *When $\omega = \sqrt{7}$, by how many radians does the output peak lag behind the input peak?*

(f) *For the forced undamped system $x'' + 9x = \cos(\omega t)$, give a detailed description of the phase lag for different input frequencies?*

Solution: (a) Natural frequency = frequency of unforced, undamped system = $\sqrt{9} = 3$.

(b) (Model solution)

Using the sinusoidal response formula: $P(r) = r^2 + 2r + 9 \Rightarrow P(i\omega) = 9 - \omega^2 + 2i\omega$. So, $|P(i\omega)| = \sqrt{(9-\omega^2)^2 + 4\omega^2}$ and $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{2\omega}{9-\omega^2}\right) \text{ in Q1 or Q2.}}$

Thus,

$$\boxed{x_p(t) = \frac{1}{|P(i\omega)|}\cos(\omega t - \phi(\omega)) = \frac{\cos(\omega t - \phi(w))}{\sqrt{(9-\omega^2)^2 + 4\omega^2}}}.$$

Alternatively (you should know how to do this):

Complexify: $z'' + 2z' + 9z = e^{i\omega t}$, $x = \operatorname{Re} z$.

Char. polynomial: $P(r) = r^2 + 2r + 9 \Rightarrow P(i\omega) = 9 - \omega^2 + 2i\omega$.

ERF: $z_p(t) = \dfrac{e^{i\omega t}}{P(i\omega)} = \dfrac{e^{i\omega t}}{9 - \omega^2 + 2i\omega}$

Polar form: $9 - \omega^2 + 2i\omega = \sqrt{(9-\omega^2)^2 + 4\omega^2}\, e^{i\phi(\omega)}$, where $\boxed{\phi(\omega) = \tan^{-1}(2\omega/(9-\omega^2)), \quad \text{in Q1 or Q2.}}$

$$\Rightarrow z_p(t) = \frac{1}{\sqrt{(9-\omega^2)^2 + 4\omega^2}} e^{i(\omega t - \phi(\omega))} \Rightarrow \boxed{x(t) = \frac{1}{\sqrt{(9-\omega^2)^2 + 4\omega^2}} \cos(\omega t - \phi(\omega))}.$$

(c) Amplitude response $= g(\omega) = \dfrac{1}{\sqrt{(9-\omega^2)^2 + 4\omega^2}}$.

(d) Practical resonance occurs if $g(\omega)$ has a local maximum. Simple calculus shows it has a maximum at $\omega = \sqrt{7}$.

(e) $\omega = \sqrt{7} \Rightarrow \phi(\omega) = \tan^{-1}\left(\dfrac{2\sqrt{7}}{9-7}\right) = \tan^{-1}\sqrt{7} \approx 1.2$ radians. The output lags behind the input by approximately 1.2 radians. (You should graph input and response on same axes.)

(f) The phase lag at ω is given by $\operatorname{Arg}(P(i\omega))$. We have

$$P(i\omega) = 9 - \omega^2 \Rightarrow \phi(\omega) = \operatorname{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } \omega < 3 \\ \pi & \text{if } \omega > 3. \end{cases}$$

Since there is no sinusoidal response when $\omega = 3$, there is no official phase lag, but the solution $x_p(t) = t\cos(3t - \pi/2)/6$ suggests that $\pi/2$ might be a reasonable unofficial choice.

Problem 7. *Consider the driven first-order system: $x' + kx = kF_0 \cos(\omega t)$.*

Consider the input to be $F_0 \cos(\omega t)$. Solve the DE and find the amplitude response. Show there is never practical resonance.

Solution: Use the sinusoidal response formula: $x_p(t) = \dfrac{kF_0 \cos(\omega t - \phi(\omega))}{|P(i\omega)|}$.

Char. polynomial: $P(r) = r + k$. So, $P(i\omega) = i\omega + k$.

Polar form: $P(i\omega) = k + i\omega \Rightarrow \boxed{|P(i\omega)| = \sqrt{k^2 + \omega^2}}$, and $\boxed{\phi(\omega) = \operatorname{Arg}(P(i\omega)) = \tan^{-1}(\omega/k), \quad \text{in Q1.}}$
So,

$$\boxed{x_p(t) = \frac{kF_0}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi(\omega))}$$

Amplitude response $= g(\omega) = \dfrac{k}{\sqrt{k^2 + \omega^2}}$. This is a decreasing function in ω, so there is no positive local maximum and no practical resonance.

End of practice quiz solutions