ES.1803 Practice Solutions – Quiz 4, Spring 2024

Problem 1.

Define all the following sets.

(i) $S_1 = \{C_1 e^{-t} + C_2 \cos(5t)\}$

(ii) $S_2 = \{C_1(e^{-t} + e^{-7t}) + C_2(e^{-t} - e^{-7t})\}$

(iii) $S_3 = \{e^{-t}, e^{-7t}\}$

(iv) Let A *and* B *both be* $m \times n$ *matrices. Let*

$$S_4 = \{\textit{All pairs of vectors } (\mathbf{x}, \mathbf{y}) | A\mathbf{x} = B\mathbf{y}\}$$

(v) $S_5 =$ *the span of the rows of a matrix* A.

(vi) $S_6 = \left\{ \begin{bmatrix} 3 \\ x+y \end{bmatrix} \right\}$

(vii) $S_7 = \left\{ \begin{bmatrix} x-3 \\ x+3 \end{bmatrix} \right\}$

(viii) $S_8 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} | x \geq 0, y \geq \right\} 0$

(a) *Which of the sets are closed under addition.*

Solution: S_1, S_2, S_4, S_5, S_8 are closed under addition; S_3, S_6, S_7 are not.

(b) *Which of the sets are closed under scalar multiplication.*

Solution: S_1, S_2, S_4, S_5 are closed under addition; S_3, S_6, S_7, S_8 are not.

(c) *Which of the sets are vector spaces.*

Solution: S_1, S_2, S_4, S_5 are closed under both addition and scalar multiplication. Hence they are vector spaces. S_3, S_6, S_7, S_8 are not.

Problem 2.

Let $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) *Find the reduced echelon form of* A.

Solution: Row reduction gives:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 = -R_2/2} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) *What is the rank of* A*?*

Solution: Two pivots implies rank $= 2$.

(c) *Find a basis for the null space of* A.

Solution: The second and fourth variables are free. Setting them to 1 and 0 in turn gives a basis

As usual, we do the computation by thinking of matrix multiplication as a linear combination of the columns, i.e.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_2 \end{bmatrix} = \mathbf{0} \Leftrightarrow x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

Then we write the variables and homogeneous solutions below each column:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -1 & 1 \end{matrix}$$

So the basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. There are, of course, many other bases, this is the one we are lead to by our standard algorithm.

(d) *Find a basis for the column space of A.*

Solution: Row reduction does not change the linear relations among the columns. In the RREF the two pivot columns form a basis of the column space of the RREF . Therefore, the corresponding columns form a basis of the column space of A: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(e) *Find a matrix with the same reduced echelon form but such that* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *are in its column space.*

Solution: The same RREF means the same null space, i.e., the same relationship between the columns. We found these relationships in Part (c), i.e., Columns 1 and 3 are pivot columns and

$$\text{Col}_2 = 2 \times \text{Col}_1, \quad \text{Col}_4 = 3 \times \text{Col}_1 + \text{Col}_3.$$

So we put the given columns as pivot columns and construct the free columns from these relationships: $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$

Note: you could put any other basis for the subspace generated by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the first columns and adjust the free columns accordingly.

Problem 3.

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

(a) *Find a basis for Null(A)*

Solution: The RREF for A is $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ We use the usual format for computing a basis of Null(A).

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} x_1 & x_2 & x_3 \\ 1 & -2 & 1 \end{matrix}$$

Thus a basis of Null(A) is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) *Is A invertible?*

No. Since Null(A) is nontrivial, A does not have an inverse.

(c) *Use Part (a) to give one eigenvalue and corresponding eigenvector of A.*

Solution: Since Null(A) is nontrivial, $\lambda = 0$ is an eigenvector and the basis vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

(d) *Find the other two eigenvalues of A.*

Solution: Since we know $\lambda = 0$ is one eigenvalue, we know the characteristic polynomial will have the form $a\lambda(\lambda^2 + b\lambda + c)$. So finding the other roots will be easy.

Finding $\det(A - \lambda)$ in a straightforward way is not hard. To make life a little easier (maybe), I used the following string of row and column operations to compute $\det(A - \lambda I)$. None of them change the determinant

$$\begin{bmatrix} 1-\lambda & 4 & 7 \\ 2 & 5-\lambda & 8 \\ 3 & 6 & 9-\lambda \end{bmatrix} \xrightarrow{C_3 = C_3 + C_1 - 2C_2} \begin{bmatrix} 1-\lambda & 4 & -\lambda \\ 2 & 5-\lambda & 2\lambda \\ 3 & 6 & -\lambda \end{bmatrix} \xrightarrow{\substack{R_1 = R_1 - R_3 \\ R_2 = R_2 + 2R_3}} \begin{bmatrix} -2-\lambda & -2 & 0 \\ 8 & 17-\lambda & 0 \\ 3 & 6 & -\lambda \end{bmatrix}$$

The determinant of the last matrix is easy to compute: $-\lambda(\lambda^2 - 15\lambda - 18)$.

So the other two eigenvalues are $\lambda = \dfrac{15 \pm \sqrt{297}}{2}$.

Problem 4.

(a) *Consider the matrix* $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$. *For which values of a is the matrix* $A - aI$ *singular? (Singular means it doesn't have an inverse.)*

Solution: $A - aI = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 6-a & 5 \\ 1 & 2-a \end{bmatrix}$. This is singular if it's determinant is 0. That is,

$$\det(A - aI) = (6-a)(2-a) - 5 = a^2 - 8a + 7 = 0$$

This happens when $a = 7$ or $a = 1$. We used the letter a, but, of course, we are just finding roots of the characteristic equation and we have found the eigenvalues of A.

(b) *For each value of a in Part (a): find the null space of $A - aI$.*

Solution: When $a = 7$, $A - 7aI = \begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix}$. Putting this in RREF we get $R = \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$.
A basis for the null space $= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. So, Null(A-7I) $= \left\{ C \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$.

When $a = 1$, $A - I = \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}$. Putting this in RREF we get $R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. A basis for the null space $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So, Null(A-I) $= \left\{ C \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

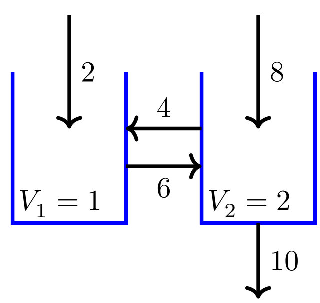
(These are eigenspaces for A.)

Problem 5.
Consider the system $x' = 5x - 6z, \quad y' = 2x - y - 2z, \quad z' = 4x - 2y - 4z.$

Rewrite this system of DEs in matrix form $\mathbf{x}' = A\,\mathbf{x}$.

Solution: $\mathbf{x}' = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \mathbf{x}$

Problem 6.
Suppose we have a two compartment system with flow rates and volumes (in some compatible units) as shown. Suppose the concentration of solute in the inflows are a and b for Tanks 1 and 2 respectively.



(a) *Give the system of DEs modeling the amounts of solute $x(t)$, $y(t)$ in Tanks 1 and 2. Write your answer in matrix form.*

Solution: The system is in balance, i.e., the volume of fluid in each compartment stays constant. The system is

$$\begin{aligned} x' &= -6\frac{x}{V_1} + 4\frac{y}{V_2} + 2a = -6x + 2y + 2a \\ y' &= 6\frac{x}{V_1} - 14\frac{y}{V_2} + 8b \ = 6x - 7y + 8b \end{aligned}$$

(In the above we made use of the values $V_1 = 1$, $V_2 = 2$.) Writing this in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 8b \end{bmatrix} \quad \text{or } \mathbf{x}' = A\mathbf{x} + \mathbf{K}.$$

(b) *Find a particular solution* $\mathbf{x_p}$ *to this inhomogeneous DE by guessing a* **constant** *solution. (Give your answer in terms of a and b.)*

Solution: The problem says to try a constant solution $\mathbf{x} = \mathbf{C}$. Substituting in the DE gives

$$\mathbf{C} = -A^{-1}\mathbf{K} = -\frac{1}{30}\begin{bmatrix} -7 & -2 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} 2a \\ 8b \end{bmatrix} = \boxed{\frac{1}{30}\begin{bmatrix} 14a + 16b \\ 12a + 48b \end{bmatrix}}.$$

End of practice quiz solutions