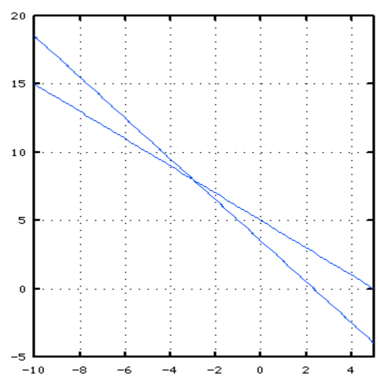
ES.1803 Practice Solutions – Quiz 4b, Spring 2024

Problem 1.
Solve this system of linear equations. How many methods can you think of to solve this system?

$$x + y = 5$$
$$3x + 2y = 7$$

Solution: Some ideas:
(1) Graphically with interesecting lines.
(2) Elimination.
(3) Row reduce the augmented matrix.
(4) Matrix inverse.

(1) $y = -x + 5$ and $y = \frac{7}{2} - \frac{3}{2}x$ are two straight lines of different slopes; so they meet at a single point. To find where, we could eyeball the picture—maybe $(-3, 8)$? That satisfies both equations!



(2) We can use elimination: Subtract 3 times the first equation from the second. Retaining the first equation as well, we get

$$x + y = 5$$
$$0 - y = -8$$

and then the first equation gives $x = -3$. In fact, as a second step, we could add the new second equation to the first one:

$$x + 0 = -3$$
$$0 - y = -8$$

Thus $(x, y) = (-3, 8)$ is the solution.

(3) Matrix methods: The system is $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

Again $(x, y) = (-3, 8)$ is the solution.

Problem 2. *Let* $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and suppose R is the reduced row echelon form for A.*

(a) *What is the rank of A?*

(b) *Find a basis for the null space of A.*

(c) *Suppose the column space of A has basis* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. *Find a possible matrix for A. That is, give a matrix with RREF R and the given column space.*

(d) *Find a matrix with the same reduced echelon form but such that* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *are in its column space.*

Solution: (a) A and R have the same rank. R has 2 pivots, so rank $= 2$.

(b) A and R have the same null space. The second and fourth variables are free. Setting them to 1 and 0 in turn gives a basis. I organize the computation in rows below the matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -1 & 1 \end{matrix}$$

So the basis consists of the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. (There are of course many other bases, this is the one we are lead to by our standard algorithm.)

(c) Looking at R, Columns 1 and 3 are pivot columns. We put the given basis in those columns:

$$A = \begin{bmatrix} 1 & * & 3 & * \\ 1 & * & 1 & * \\ 0 & * & 1 & * \end{bmatrix}$$

The free columns of R are linear combinations of the pivot columns and those of A are the same linear combinations. In R it is clear that

$$\text{Col}_2 = 2 \times \text{Col}_1 \text{ and } \text{Col}_4 = 3 \times \text{Col}_1 + \text{Col}_3.$$

So,

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(d) We found the relationships between the columns in Part (c). So we put the given columns as pivot columns and construct the free columns from these relationships: $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$

Note: you could put any other basis for the subspace generated by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the first columns and adjust the free columns accordingly.

Problem 3. *Consider the following system of equations:*

$$\begin{aligned} x + \ \ y + \ \ z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) *Write this system of equations as a matrix equation.*

(b) *Use row reduction to get to row echelon form. What is the solution set?*

Solution: (a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

(b) Set up the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \\ 1 & 3 & 6 & 11 \end{array}\right]$.

We use our usual notation for rows, e.g., Row $2 = R_2$. Here is the sequence of row operations leading to the RREF of the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \\ 1 & 3 & 6 & 11 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 5 & 6 \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - 2R_3]{R_1 = R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The solution is $x = 5, y = -2, z = 2$. You can check this by substituting it into the original equation.

Problem 4.

(a) *Try to solve the following equation using row reduction:*

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

At the end of the row reduction process, was the augmented column pivotal or free? Is this related to the absence of solutions?

Solution: We augment the coefficient matrix and do row reduction:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -3 \end{array}\right]$$

At this point, we're in trouble because the bottom row represents the equation

$$0x + 0y = -3.$$

Clearly this cannot be solved. There is no need to continue to the RREF.

Yes, the last column is a pivot column. As we saw, in reduced form this means there are all zeros to the left of the pivot. This implies the equation

$$0 = \text{nonzero},$$

which has no solutions.

(b) *Find a new vector* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *such that* $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *has a solution. What is the set of all solutions of this new equation?*

Solution: Well, we could always take $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because the equation is then obviously solved by $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

But we can also take $\mathbf{b}$ to be the first column of the coefficient matrix, i.e., $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. A solution is then $\mathbf{x_p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (More generally, we can let $\mathbf{b}$ be any vector in the column space of the matrix.)

The general solution to $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is given by a particular solution plus the general homogeneous solution. We know a particular solution is $\mathbf{x_p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The null space is $c \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. (See the row reduction in Part (a).) So the general solution to the equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Problem 5. *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 0 & 0 & 10 & 3 & 6 \end{bmatrix}$. *Put* A *in row reduced echelon form. Find the rank, a basis of the column space, a basis of the null space, and the dimension of each of the spaces.*

Solution: Here are the row reduction steps:

$$A \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 3 & 6 \end{bmatrix} \xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 10 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2/10} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3/10 & 6/10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & 1/10 & 2/10 \\ 0 & 0 & 1 & 3/10 & 6/10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The pivot columns are Columns 1 and 3. These give a basis for the column space of A.

$$\text{Basis: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\} \qquad \text{Col}(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\}$$

Rank(A) = # of pivots = dimension of column space = 2.

The null space of A has dimension 3 = the number of free variables. Since A and R have the same null space, we work with R.

We find the basis by setting, in turn, each free variable to 1 and the others to 0 and then solve for the pivot variables. We do the computation by putting the values below the RREF matrix R.

$$\begin{bmatrix} 1 & 2 & 0 & 1/10 & 2/10 \\ 0 & 0 & 1 & 3/10 & 6/10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1	x_2	x_3	x_4	x_5
-2	1	0	0	0
$-1/10$	0	$-3/10$	1	0
$-2/10$	0	$-6/10$	0	1

The basis is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/10 \\ 0 \\ -3/10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/10 \\ 0 \\ -6/10 \\ 0 \\ 1 \end{bmatrix}$. Or we could use, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -6 \\ 0 \\ 10 \end{bmatrix}$.

Problem 6. **(a)** *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & x \end{bmatrix}$$

Can you specify x?

Solution: Each column of the product is a multiple of the column vector in the first factor. The 1 and 3 in the top row show that the second column is 3 times the first. So x must be 6.

(b) *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet & 3 \\ 4 & \bullet \\ \bullet & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the •'s?

Solution: The matrix equation says: the first column plus twice the second column is zero, i.e, $\text{Col}_1 = -2\text{Col}_2$. So the matrix must be $\begin{bmatrix} -6 & 3 \\ 4 & -2 \\ -12 & 6 \end{bmatrix}$.

Problem 7. *Suppose we have a matrix equation* $A = \begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix}$. *All we know about A is that Null(A) is nontrivial. What can we say about* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$*?*

Solution: A nontrivial null space implies the columns are not independent. Since Columns 1 and 3 are independent, we must have that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a linear combination of Columns 1 and 3. Geometrically, the point (x, y, z) lies on the plane in $\mathbf{R}^3$ spanned by these columns.

Problem 8. *For what values of y, z are the columns of the matrix* $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & y \\ 5 & 6 & z \end{bmatrix}$ *linearly independent?*

Solution: The columns are linearly independent when the matrix has rank 3. We can find the rank by row reduction:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & y \\ 5 & 6 & z \end{bmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & y-6 \\ 0 & 1 & z-10 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & y-6 \\ 0 & 0 & z-y-4 \end{bmatrix}$$

If $z - y - 4 \neq 0$, then we have 3 pivots. So the columns are linearly independent exactly when $z - y \neq 4$.

Problem 9. *Let* $A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$*:*

(a) *Find the row reduced echelon form of A; call it R.*

Solution: Here are the row reduction steps:

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -10 & -20 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow{R_2 = -R_2/10} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 4R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

(b) *The last column of R should be a linear combination of the first columns in an obvious way. Find a vector $\mathbf{x}$, such that $R\mathbf{x} = \mathbf{0}$, which expresses this linear relationship.*

Solution: This is just an awkward way of asking about null vectors. The third column is free. By inspection of R, we see that

$$\text{Col}_3 = -\text{Col}_1 + 2\text{Col}_2 \quad \Rightarrow \text{Col}_1 - 2\text{Col}_2 + \text{Col}_3 = \mathbf{0}.$$

So, $R \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) *Verify that the same relationship holds among the columns of A.*

Solution: The third column is indeed twice the second minus the first. As a matrix equation,

$$A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 10. *This continues the previous problem. Now suppose we want to solve $A\mathbf{x} = \mathbf{b}$.*

(a) *For what $\mathbf{b}$ is this possible?*

Solution: The equation can be solved provided that $\mathbf{b}$ is in the column space of A. From the previous problem, we know that Columns 1 and 2 are the pivot columns. So, Columns 1 and 2 give a basis of Col(A). That is

$$\text{Col}(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

These are all the $\mathbf{b}$ for which the equation has a solution.

(b) *For those $\mathbf{b}$ found in Part (a), describe the general solution to $A\mathbf{x} = \mathbf{b}$.*

Solution: As always, the general solution is particular plus homogeneous. The previous problem told us that Null(A) is one dimensional and gave us a basis. So, the homogeneous solution is

$$\mathbf{x_h} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The nice thing about writing $\mathbf{b}$ as a linear combination of columns is that it hands us a particular solution. That is, if $\mathbf{b} = c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has solution $\mathbf{x_p} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$.

The general solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 11. *Suppose that the reduced echelon form of the 4×6 matrix B is*

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) *Find a basis for Null(B).*

Solution: Since Null(B) = Null(R), we use R to find a basis. As usual, we do this by setting each free variable to 1 in turn. The free variables are x_1, x_3, x_4, x_6. In our usual format:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 & -6 & 1 \end{array}$$

A basis for Null(B) is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix}$.

(b) *Suppose we write $B = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} & \mathbf{b_4} & \mathbf{b_5} & \mathbf{b_6} \end{bmatrix}$, where $\mathbf{b_1}$ etc. are the columns? What can you say about these columns.*

Solution: All we know from R are the relations between the columns.

So, $\mathbf{b_2}$, $\mathbf{b_5}$ can be any two linearly independent vectors in $\mathbf{R}^4$.

$\mathbf{b_1} = \mathbf{0}$.

$\mathbf{b_3} = 2\mathbf{b_2}, \quad \mathbf{b_4} = 3\mathbf{b_2}, \; \mathbf{b_6} = 5\mathbf{b_2} + 6\mathbf{b_5}$.

End of practice quiz solutions