ES.1803 Practice Solutions – Quiz 4c, Spring 2024

Problem 1

Consider the system $x' = -3x + 2y\,, \quad y' = -x - y.$

Find the solution $x(t)$, $y(t)$ *satisfying the IC's* $x(0) = 0$, $y(0) = 1$.

Solution: Coefficient matrix: $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$

Chararacteristic equation: $\det \begin{bmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 5 = 0.$

Eigenvalues (roots): $\lambda = -2 \pm i$. (Complex values.)

Basic eigenvectors (basis of $\text{Null}(A - \lambda I)$):

$\lambda = -2 + i: \quad A - \lambda I = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$. Basic eigenvector $\begin{bmatrix} 2 \\ 1+i \end{bmatrix}$.

We get two solutions from one complex-valued solution by taking the real and imaginary parts. This requires a bit of complex arithmetic.

Complex solution: $\mathbf{z}(t) = e^{(-2+i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$. Expanding this we get

$$\begin{aligned}
\mathbf{z}(t) &= e^{(-2+i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} \\
&= e^{-2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 2 \\ 1+i \end{bmatrix} \\
&= e^{-2t} \begin{bmatrix} 2\cos(t) + i2\sin(t) \\ \cos(t) - \sin(t) + i(\cos(t) + \sin(t)) \end{bmatrix} \\
&= e^{-2t} \begin{bmatrix} 2\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix} + ie^{-2t} \begin{bmatrix} 2\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}
\end{aligned}$$

Taking real and imaginary parts, we have two (real-valued) solutions:

$$\mathbf{x_1(t)} = e^{-2t} \begin{bmatrix} 2\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix}, \qquad \mathbf{x_2(t)} = e^{-2t} \begin{bmatrix} 2\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}$$

The general solution is then

$$\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 e^{-2t} \begin{bmatrix} 2\cos t \\ \cos t - \sin t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2\sin t \\ \cos t + sint \end{bmatrix}.$$

We use the initial condition to find c_1 and c_2:

$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow c_1 = 0,\ c_2 = 1.$ So, $\boxed{\mathbf{x(t)} = e^{-2t} \begin{bmatrix} 2\sin t \\ \cos t + \sin t \end{bmatrix}}$.

Problem 2

Consider the system $x' = 5x - 6z, \quad y' = 2x - y - 2z, \quad z' = 4x - 2y - 4z.$

(a) *Rewrite this system of DEs in matrix form* $\mathbf{x}' = A\,\mathbf{x}$.

Solution: $\mathbf{x}' = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \mathbf{x}$

(b) *Call the coefficient matrix A. Given that the eigenvalues of A are 0, -1 and 1, write down the form of the three normal modes $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$ for this system without solving the system explicitly, i.e., just give the eigenvectors names, but don't find them.*

Solution: Let $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ be basic eigenvectors corresponding to 0, -1 and 1 respectively. The three normal modes are: $\mathbf{x_1}(t) = \mathbf{v_1}$, $\mathbf{x_2}(t) = e^{-t}\mathbf{v_2}$ and $\mathbf{x_3}(t) = e^{t}\mathbf{v_3}$.

(c) *What is the long-run behavior (i.e., as $t \to \infty$) of the general solution to this system of DEs ? Justify your answer (briefly).*

Solution: Long run means as $t \to \infty$. The main point is that in the long term $e^{-t} \to 0$ and $e^t \to \infty$.

Therefore, $\mathbf{x_1}(t)$ is constant, $\mathbf{x_2}(t)$ goes to 0 and $\mathbf{x_3}(t)$ goes to ∞ in the $\mathbf{v_3}$ direction.

Since, the general solution is $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) + c_3\mathbf{x_3}(t)$.:

If $c_3 \neq 0$ then $\mathbf{x}(t)$ goes to ∞ in the $\mathbf{v_3}$ direction.

If $c_3 = 0$ then $\mathbf{x}(t)$ goes to $c_1\mathbf{v_1}$.

(d) *Find the eigenvectors and give the explicit solution to the system.*

Solution: To find eigenvectors, we must find bases for Null$(A - \lambda I)$. As usual, we do this by first finding the RREF, then setting the free variable to 1 and solving for the pivot variables.

$$\lambda = 0: \; A - \lambda I = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \xrightarrow{R_1 = R_1/5} \begin{bmatrix} 1 & 0 & -6/5 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -6/5 \\ 0 & -1 & 2/5 \\ 0 & -2 & 4/5 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -6/5 \\ 0 & -1 & 2/5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = -R_2} \begin{bmatrix} 1 & 0 & -6/5 \\ 0 & 1 & -2/5 \\ 0 & 0 & 0 \end{bmatrix},$$

So, a basic null vector is $\begin{bmatrix} 6/5 \\ 2/5 \\ 1 \end{bmatrix}$. Even better, we take $\boxed{\mathbf{v_1} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}}$.

$$\lambda = -1: \; A - \lambda I = \begin{bmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix} \xrightarrow{R_1 = R_1/6} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix} \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = -R_2/2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, a basic null vector is $\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$. Even better, we take $\boxed{\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$.

$$\lambda = 1:\ A - \lambda I = \begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix} \xrightarrow{R_1 = R_1/4} \begin{bmatrix} 1 & 0 & -3/2 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix} \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = -R_2/2} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, a basic null vector is $\begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$. Even better, we take $\boxed{\mathbf{v_3} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}$.

The general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(e) *Diagonalize the coefficient matrix.*

Solution: We know $A = S\Lambda S^{-1}$, where S is the matrix of eigenvectors and Λ is the diagonal matrix of corresponding eigenvalues. From Part (d)

$$S = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 1 & 1 \\ 5 & 2 & 2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 3 & 0 \\ 1 & 2 & -2 \end{bmatrix}$$

We found S^{-1} by Laplace's method. (Unless we ask for a specific method, any one will do.)

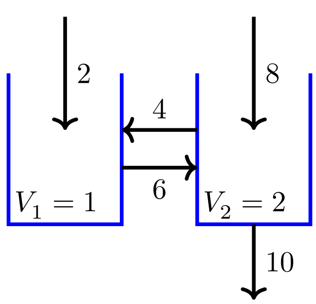
(f) *Decouple the system.*

Solution: Decoupling means changing variables: $\mathbf{x} = S\mathbf{u}$ which gives the diagonal system $\mathbf{u}' = \Lambda \mathbf{u}$. In this case, we have

$$\mathbf{x} = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 1 & 1 \\ 5 & 2 & 2 \end{bmatrix} \mathbf{u}, \qquad \mathbf{u}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u}$$

Problem 3

On Practice quiz 4 we had the following two compartment system with flow rates and volumes (in some compatible units) as shown. The concentration of solute in the inflows are constants a and b for Tanks 1 and 2 respectively.



Let x and y be the amounts of solute in tanks 1 and 2. In matrix form the system is modeled by

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -6 & 2 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 8b \end{bmatrix} \quad \text{or } \mathbf{x}' = A\mathbf{x} + \mathbf{K}.$$

In Practice quiz 4 we also found that the system has a constant solution

$$\mathbf{x_p}(t) = \frac{1}{30} \begin{bmatrix} 14a + 16b \\ 12a + 48b \end{bmatrix}.$$

Show that the solution $\mathbf{x}_p$ is the "steady-state" solution for this system, in the sense that all solutions to this system approach this particular solution $\mathbf{x_p}$ as t goes to ∞. (Note: this can be done without lots of calculation.)

Solution: We only need to show that the eigenvalues of A are negative. That is, if the eigenvalues are negative, then the homogeneous part of the solution goes to 0 no matter what the initial conditions.

Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 2 \\ 6 & -7-\lambda \end{vmatrix} = \lambda^2 + 13\lambda + 30 = 0 \Rightarrow \lambda = -10, -3$. QED

(Note: we didn't really need to find the roots because we know that for a quadratic equation if all the coefficients are positive then the roots all have negative real part.)

Problem 4

Let $A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix}$

(a) *What are the eigenvalues of A?*

Solution: Characteristic equation: $\lambda^2 - 4\lambda - 32 = 0$. So, $\lambda = 8, -4$.

(b) *For each eigenvalue, find a basic eigenvector.*

Solution: $\lambda = 8$: $(A - \lambda I) = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda = -4$: $(A - \lambda I) = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Problem 5

Suppose that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively.

What is the solution to $\mathbf{x}' = B\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Solution: The general solution is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We use the initial conditions to solve for c_1 and c_2.

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{i.e.,} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solving (any way you want) gives $c_1 = 3/2$, $c_2 = -1/2$, so $\boxed{\mathbf{x}(t) = \frac{3}{2}e^t\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}e^{2t}\begin{bmatrix}-1\\1\end{bmatrix}}$.

Problem 6

Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $\mathbf{x}' = A\mathbf{x}$. Determine all values of a (if any) for which the system is stable

Solution: The key observation is the same as before: the exponents in solutions must be negative or have negative real part. That is, the eigenvalues must be negative or have negative real part.

First, note that the characteristic equation is $\lambda^2 - (a+1)\lambda + a + 4$.

The easiest way to decide about stability is to recall that the roots of a quadratic have negative real part only if all the coefficients have the same sign. Since the leading coefficient is 1, this means that we must have $-(a+1) > 0$ and $a + 4 > 0$. So, $\boxed{-4 < a < -1}$.

We've ignored the case of pure imaginary roots which is an edge case and is sometimes called stable, but not 'asymptotically stable'.

Note: if you wanted to, you could write out the roots and fight through the algebra. The quadratic equation gives roots

$$\lambda = \frac{a + 1 \pm \sqrt{(a+1)^2 - 4(a+4)}}{2} = \frac{a + 1 \pm \sqrt{a^2 - 2a - 15}}{2}$$

In the end, you will conclude that the roots have negative real part when $-4 < a < -1$.

Problem 7

Convert the following to a system of DEs and use matrix methods to find a solution for $x(t)$

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

Solution: We used the physicists notation of dots for time derivatives. This problem is a little silly, since the characteristic equations of the DE and the system are the same.

The companion system is

$$\dot{x} = y; \ \dot{y} = -2x - 2y \quad \Leftrightarrow \quad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

Chararacteristic equation: $\lambda^2 + 2\lambda + 2$.

Eigenvalues: $\lambda = -1 \pm i$.

Basic eigenvectors:

$\lambda_1 = -1 + i$: $\quad (A - \lambda I) = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$. So, a basic eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$.

(Complex) mode: $\mathbf{z}(t) = e^{(-1+i)t}\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$.

Expand and find real and imaginary parts:

$$\mathbf{z}(t) = e^{-t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = e^{-t} \begin{bmatrix} \cos(t) + i\sin(t) \\ -\cos(t) - \sin(t) + i(\cos(t) - \sin(t)) \end{bmatrix}.$$

General (real) solution: $\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos(t) \\ -\cos(t) - \sin(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin(t) \\ \cos(t) - \sin(t) \end{bmatrix}$.

Finally, we just take the x-component of the solution: $\boxed{x(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t).}$

Problem 8

Let $A = \begin{bmatrix} a & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

(a) *What is the determinant of A?*

Solution: This is triangular, so $\det(A) = $ diagonal entries $= 15a$.

(b) *What are the eigenvalues of A?*

Solution: For an upper triangular matrix the eigenvalues are the diagonal entries: a, 3, 5.

(c) *For what value (or values) of a is A singular (non-invertible)?*

Solution: $\det(A) = 15a$. So A is singular when $a = 0$.

(d) *What is the minimum rank of A (as a varies)? What's the maximum?*

Solution: When $a = 0$, the null space is dimension 1, so rank =2.

When $a \neq 0$, A invertible, i.e, rank = 3.

Problem 9

Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$. *Where S is an invertible* 3×3 *matrix.*

(a) *What are the eigenvalues of A?*

Solution: A is given in diagonalized form, so the eigenvalues are the entries in the diagonal matrix, i.e., 1, 2, 3.

(b) *Express* A^2, A^{-1} *in terms of S.*

Solution: Call the diagonal matrix matrix in the problem D, so $A = SDS^{-1}$. Then

$$A^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1} = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} S^{-1}$$

$$A^{-1} = (SDS^{-1})^{-1} = SD^{-1}S^{-1} = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} S^{-1}.$$

(c) *For the system* $\mathbf{x}' = A\mathbf{x}$, *is the equilibrium at the origin stable, unstable, or neither?*

Solution: There are positive eigenvalues, so the system is unstable.

(d) *What would I need to know about S in order to write down the most rapidly growing exponential solution to* $\mathbf{x}' = A\mathbf{x}$*?*

Solution: You need to know the eigenvector that goes with the eigenvalue 3. That is, you need to know the third column of S.

Problem 10

(a) *Find an orthogonal matrix S ($S^T S = I$) and a diagonal matrix Λ such that $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = S\Lambda S^{-1}$ Don't worry about the term orthogonal, we didn't do it in class. The problem just asks you to diagonalize the matrix. Notice that the eigenvectors are perpendicular to each other*

Solution: An orthogonal matrix has columns that are orthonormal (mutually orthogonal and unit length). It is a fact that a symmetric matrix has mutually orthogonal eigenvectors.

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

A has characteristic equation: $\lambda^2 - 4\lambda + 3$ and eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$. By inspection (or computation) we have eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These are clearly orthogonal to each other, we normalize their lengths and use the normalized eigenvectors in the matrix S.

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

(b) *Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: that is, find coordinates $u_1 = ax_1 + bx_2$, $u_2 = cx_1 + dx_2$, such that the equation is equivalent to $u_1' = \lambda_1 u_1$, $u_2' = \lambda_2 u_2$.*

Solution: The decoupling equation is $\mathbf{u} = S^{-1}\mathbf{x}$. That is $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

That is $u_1 = x_1/\sqrt{2} - x_2/\sqrt{2}$; $\quad u_2 = x_1/\sqrt{2} + x_2/\sqrt{2}$.

The DE is $\quad u_1' = u_1$; $\quad u_2' = 3u_2$

End of practice quiz solutions