ES.1803 Practice Questions – Quiz 4c, Spring 2024
Covers Topics 15-17

This will probably take longer than 1 hour; the actual quiz will be considerably shorter. Not every possible question is covered. For more questions, you can also look at the in-class problems, problem section worksheets and psets.

Be wise in carrying out computations. If you get a problem to a point where you are sure –*emphasize sure*– you can carry out the computation, then stop and go on to the next problem. At the end you can check the solutions to see that you had correctly set up the computation.

Problem 1
Consider the system $x' = -3x + 2y\,,\quad y' = -x - y.$

Find the solution $x(t)$, $y(t)$ satisfying the IC's $x(0) = 0$, $y(0) = 1$.

Problem 2
Consider the system $x' = 5x - 6z,\quad y' = 2x - y - 2z,\quad z' = 4x - 2y - 4z.$

(a) Rewrite this system of DEs in matrix form $\mathbf{x}' = A\,\mathbf{x}$.

(b) Call the coefficient matrix A. Given that the eigenvalues of A are 0, -1 and 1, write down the *form* of the three normal modes $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ for this system *without* solving the system explicitly, i.e., just give the eigenvectors names, but don't find them.

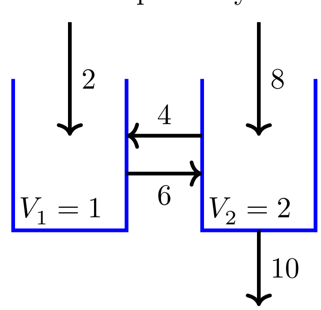
(c) What is the long-run behavior (i.e., as $t \to \infty$) of the general solution to this system of DEs ? Justify your answer (briefly).

(d) Find the eigenvectors and give the explicit solution to the system.

(e) Diagonalize the coefficient matrix.

(f) Decouple the system.

Problem 3
On Practice quiz 4 we had the following two compartment system with flow rates and volumes (in some compatible units) as shown. The concentration of solute in the inflows are constants a and b for Tanks 1 and 2 respectively.



Let x and y be the amounts of solute in tanks 1 and 2. In matrix form the system is modeled by

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -6 & 2 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 8b \end{bmatrix} \quad \text{or } \mathbf{x}' = A\mathbf{x} + \mathbf{K}.$$

In Practice quiz 4 we also found that the system has a constant solution

$$\mathbf{x_p}(t) = \frac{1}{30}\begin{bmatrix} 14a + 16b \\ 12a + 48b \end{bmatrix}.$$

Show that the solution $\mathbf{x}_p$ is the "steady-state" solution for this system, in the sense that all solutions to this system approach this particular solution $\mathbf{x_p}$ as t goes to ∞. (Note: this can be done without lots of calculation.)

Problem 4

Let $A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix}$

(a) What are the eigenvalues of A?

(b) For each eigenvalue, find a basic eigenvector.

Problem 5

Suppose that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively.

What is the solution to $\mathbf{x}' = B\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Problem 6

Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $\mathbf{x}' = A\mathbf{x}$. Determine all values of a (if any) for which the system is stable

Problem 7

Convert the following to a system of DEs and use matrix methods to find a solution for $x(t)$

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

Problem 8

Let $A = \begin{bmatrix} a & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

(a) What is the determinant of A?

(b) What are the eigenvalues of A?

(c) For what value (or values) of a is A singular (non-invertible)?

(d) What is the minimum rank of A (as a varies)? What's the maximum?

Problem 9

Suppose that $A = S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}S^{-1}$. Where S is an invertible 3×3 matrix.

(a) What are the eigenvalues of A?

(b) Express A^2, A^{-1} in terms of S.

(c) For the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable, unstable, or neither?

(d) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

Problem 10

(a) Find an orthogonal matrix S ($S^T S = I$) and a diagonal matrix Λ such that $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = S\Lambda S^{-1}$ Don't worry about the term orthogonal, we didn't do it in class. The problem just asks you to diagonalize the matrix. Notice that the eigenvectors are perpendicular to each other

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: that is, find coordinates $u_1 = ax_1 + bx_2$, $u_2 = cx_1 + dx_2$, such that the equation is equivalent to $u_1' = \lambda_1 u_1$, $u_2' = \lambda_2 u_2$.

End of practice quiz