ES.1803 Practice Solutions – Quiz 5, Spring 2024

Integrals (for n a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$

Some Fourier series:

1. Period 2π square wave $\text{sq}(t)$: *You should know this for the quiz.*
2. Period 2 triangle wave $\text{tri2}(t)$:

   Over one period, $-1 \le t \le 1$, $\text{tri2}(t) = |t|$.

$$\begin{aligned} \text{tri2}(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right) \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}. \end{aligned}$$

Problem 1. (Step and delta)

(a) *Compute the following integrals.*

(i) $\displaystyle\int_{-\infty}^{\infty} e^{\lambda t}(\delta(t) + 2\delta(t-1) + 3\delta(t-2))\,dt$

(ii) $\displaystyle f(t) = \int_{0^-}^{t} 2\delta(\tau+1) + 3\delta(\tau) + 4\delta(\tau-1)\,d\tau$. *Assume* $t > 0$. *What is* $f(3)$*?*

(iii) $\displaystyle f(t) = \int e^{\lambda t}(\delta(t) + 2\delta(t-1) + 3\delta(t-2))\,dt$

Solution: (i) The computation is just substitution: $1 + 2e^{\lambda} + 3e^{2\lambda}$

(ii) Because the upper limit is t, the answer should be a function of t: The spike for $\delta(\tau+1)$ is at -1, which is outside the interval of integration. The other two spikes are in the interval of integration, so $\boxed{f(t) = 3u(t) + 4u(t-1)}$; $\boxed{f(3) = 7}$.

(iii) This is an indefinite integral, so the answer will be a function of t. A key point is the following: If f is continuous, then $f(t)\delta(t-a) = f(a)\delta(t-a)$. This is because $\delta(t-a)$ is zero away from a. So,

$$e^{\lambda t}\delta(t) = \delta(t); \quad e^{\lambda t}\delta(t-1) = e^{\lambda}\delta(t-1); \quad e^{\lambda t}\delta(t-2) = e^{2\lambda}\delta(t-2).$$

So the integral is

$$\boxed{u(t) + 2e^{\lambda}u(t-1) + 3e^{2\lambda}u(t-2) + C.}$$

Note that there is no t in the exponentials.

(b) *Solve the following initial value problems.*

(i) $2\ddot{x} + 7\dot{x} + 3x = \delta(t)$ *with rest initial conditions.*

(ii) $2\ddot{x} + 7\dot{x} + 3x = \delta(t) + e^{3t}$ *with* $x(0^-) = 0$, $\dot{x}(0^-) = 0$.

Solution: (i) On $t < 0$: $\delta(t) = 0$, so the initial value problem (IVP) is

$$2\ddot{x} + 7\dot{x} + 3x = 0, \quad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

We know that this has solution $x(t) = 0$.

Case $t > 0$: The pre-initial conditions are $x(0^-) = 0$m $x'(0^-) = 0$. So the post-initial conditions are

$$x(0^+) = x(0^-) = 0, \quad \dot{x}(0^+) = \dot{x}(0^-) + \frac{1}{2} = \frac{1}{2}2.$$

Since $\delta(t)$ on $t > 0$, we need to solve the homogeneous IVP

$$2\ddot{x} + 7\dot{x} + 3x = 0, \quad x(0^+) = 0,\ \dot{x}(0^+) = \frac{1}{2}.$$

The characteristic equation $2r^2 + 7r + 3 = 0$ has roots $r = -1/2, -3$. So the general homogeneous solution is $x(t) = c_1e^{-t/2} + c_2e^{-3t}$. Now the post-initial conditions give:

$$c_1 + c_2 = 0; \quad -\frac{c_1}{2} - 3c_2 = 1/2.$$

Solving we get: $c_1 = 1/5$ and $c_2 = -1/5$. So the solution is $x(t) = e^{-t/2}/5 - e^{-3t}/5$ for $t > 0$.

Putting the two pieces together, we have: $\boxed{x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-t/2}/5 - e^{-3t}/5 & \text{for } t > 0. \end{cases}}$

(ii) Since rest IC are homogeneous, we can solve in pieces and use superposition.

Piece 1: $2\ddot{x}_1 + 7\dot{x}_1 + 3x_1 = \delta(t)$ with rest IC. This is just Part (i):

$$x_1(t) = \frac{e^{-t/2}}{5} - \frac{e^{-3t}}{5} \ \text{ for } t > 0.$$

Piece 2: $2\ddot{x}_2 + 7\dot{x}_2 + 3x_2 = e^{3t}$ with $x(0) = 0$, $\dot{x}(0) = 0$.

Particular solution (using the ERF): $\dfrac{e^{3t}}{42}$.

General solution: $x_2(t) = e^{3t}/42 + c_1 e^{-t/2} + c_2 e^{-3t}$.

Using the initial conditions, we find $c_1 = -2/35$, $c_2 = 1/30$. So,

$$x_2(t) = \frac{e^{3t}}{42} - \frac{2e^{-t/2}}{35} + \frac{e^{-3t}}{30}.$$

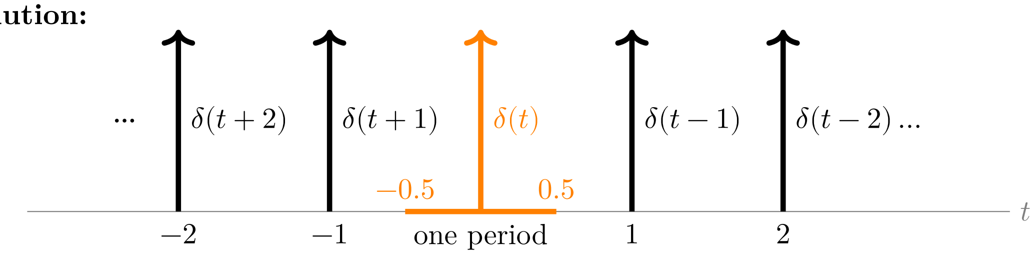
By superposition the solution is

$$\boxed{x(t) = x_1(t) + x_2(t) = \begin{cases} e^{3t}/42 - 2e^{-t/2}/35 + e^{-3t}/30 & \text{for } t < 0 \\ e^{3t}/42 + e^{-t/2}/7 - e^{-3t}/6 & \text{for } t > 0. \end{cases}}$$

(c) *Compute the Fourier series for the period 1 impulse train*

$$f(t) = \ldots + \delta(t+2) + \delta(t+1) + \delta(t) + \delta(t-1) + \ldots$$

Solution:



Since the function is even, we know $b_n = 0$. The delta function at the origin is going to make the 'doubling trick' for even functions a little tricky. Instead, we integrate over a full period. The period is 1 so $L = 1/2$ and

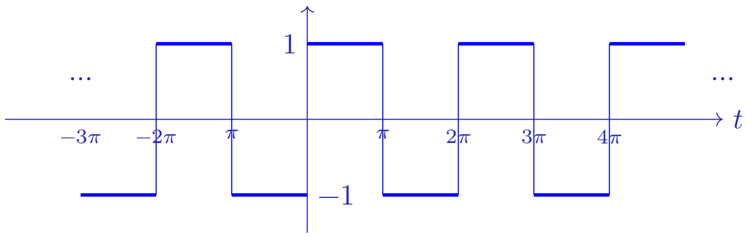
$$a_n = \frac{1}{1/2}\int_{-1/2}^{1/2} f(t)\cos(2n\pi t)\,dt = 2\int_{-1/2}^{1/2} \delta(t)\cos(2n\pi t)\,dt = 2, \quad a_0 = 2\int_{-1/2}^{1/2} f(t)\,dt = 2\int_{-1/2}^{1/2} \delta(t)\,dt = 2.$$

Therefore,

$$\boxed{f(t) = 1 + 2\sum_{n=1}^{\infty} \cos(2n\pi t).}$$

Problem 2. Derivative of a square wave

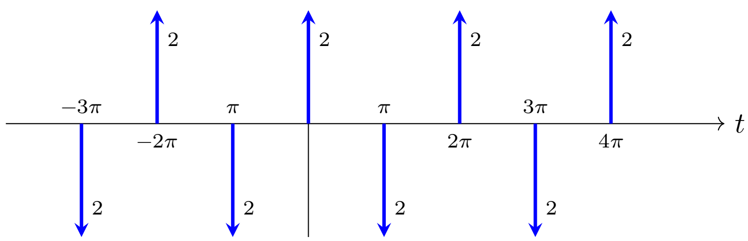
The graph below is of the function $\mathrm{sq}(t)$ *(standard square wave). Compute and graph its generalized derivative.*



Graph of $\mathrm{sq}(t)$ = *square wave*

Solution: The function alternates every π seconds between ± 1. The derivative $\mathrm{sq}'(t)$ is clearly 0 everywhere except at the jumps. A jump of $+2$ gives a (generalized) derivative of 2δ and a jump of -2 gives a (generalized) derivative of -2δ. Thus we have

$$\mathrm{sq}'(t) = ... + 2\delta(t+2\pi) - 2\delta(t+\pi) + 2\delta(t) - 2\delta(t-\pi) + 2\delta(t-2\pi) - 2\delta(t-3\pi) + ...$$



Graph of $\mathrm{sq}'(t)$ = impulse train

Note that we put the weight of each delta function next to it. We use the convention that $-2\delta(t)$ is represented by a downward arrow with the weight 2 next to it. That is, the sign is represented by the direction of the arrow, so the weight is positive.

Problem 3.

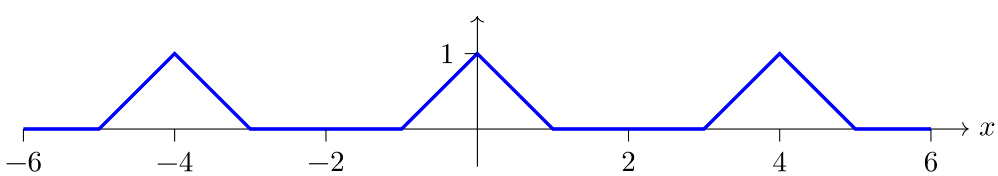
Let $f(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ 0 & 1 \le x < 2 \end{cases}$

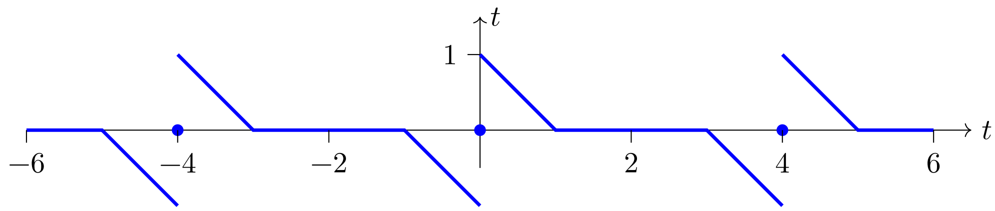
(a) *Sketch the following periodic extensions of* f *over three or more full periods:*

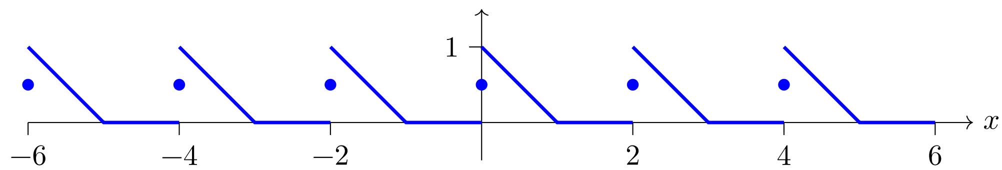
(i) even period 4 extension (ii) odd period 4 extension (iii) periodic extension with period 2.

For (ii) and (iii), also sketch in the 'extra' points to which the Fourier series expansion will converge (without computing the Fourier series).

Solution: Here (in order) are the even periodic, odd periodic and periodic extenstions of f.







(b) *Compute the Fourier cosine series of f.*

Solution: $L = 2$. So,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi}{L}nx\right)dx = \int_0^2 f(x)\cos\left(\frac{\pi}{2}nx\right)dx = \int_0^1 (1-x)\cos\left(\frac{\pi}{2}nx\right)dx.$$

$$a_0 = \int_0^1 (1-x)dx = \frac{1}{2}.$$

For $n \neq 0$, using the integral table (or integration by parts), we get

$$a_n = \frac{4\left(1-\cos\left(\frac{\pi}{2}n\right)\right)}{\pi^2 n^2}.$$

Writing it out to see the pattern: $a_n = 1-\cos(\frac{\pi}{2}n) = 0, 1, 2, 1, 0, 1, 2, 1, \ldots$. I don't see a nicer way to write these. Therefore, the cosine series is

$$\boxed{f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4(1-\cos\left(\frac{\pi}{2}n\right))}{\pi^2 n^2}\cos\left(\frac{\pi}{2}nx\right).}$$

(c) *Write out (with decimal numbers) the first 4 terms of the series in Part (b).*

Solution: $f(x) \approx 0.25 + 0.4053\cos\left(\frac{\pi}{2}x\right) + 0.2026\cos\left(\frac{\pi}{2}2x\right) + 0.0450\cos\left(\frac{\pi}{2}3x\right).$

(Note: On an exam we would make sure that any hand calculation was easier to calculate.)

(d) *Compute the steady periodic solution to the DE* $x''(t) + 2.5\,x(t) = \tilde{f}_e(t)$ *where* $\tilde{f}_e$ *is the even periodic extension of f.*

Solution: The Fourier series for $\tilde{f}_e$ is just the cosine series from Part (b). So we solve the DE with each term of $\tilde{f}_e$ as input and then use superposition. Using the SRF, the equation $x_n'' + 2.5x_n = \cos\left(\frac{\pi}{2}nt\right)$ has solution

$$x_{n,p}(t) = \frac{4}{|10-\pi^2 n^2|}\cos\left(\frac{\pi}{2}nt - \phi(n)\right),$$

where $\phi(n) = \text{Arg}((10-\pi^2n^2)/4) = \begin{cases} 0 & \text{if } n \leq 1 \\ \pi & \text{if } n \geq 2 \end{cases}$

Using superposition, the periodic-solution to the DE is

$$\boxed{x_p(t) = \frac{4}{10} \cdot \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{4a_n}{|10 - \pi^2 n^2|} \cos\left(\frac{\pi}{2} nt - \phi(n)\right)},$$

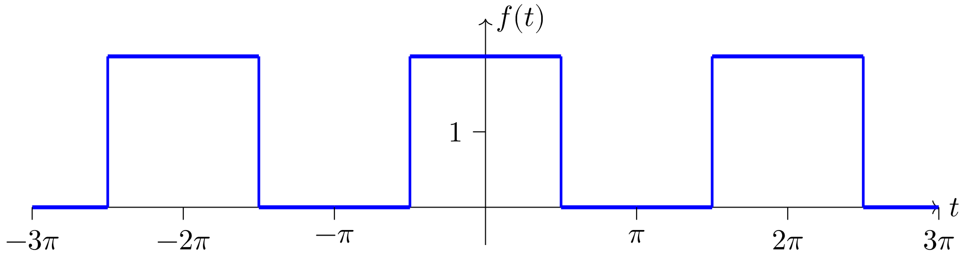
where a_n is as in Part (b).

Problem 4.

Let $\tilde{f}_e(t)$ be the even period 2π extension to the function $f(t) = \begin{cases} 2 & \text{if } 0 < t < \pi/2 \\ 0 & \text{if } \pi/2 < t < \pi. \end{cases}$

Solve $\dot{x} + kx = \tilde{f}_e(t)$ for the periodic solution in Fourier series form,

Solution: First we plot $\tilde{f}_e(t)$, which has period 2π.



The Fourier series for $\tilde{f}_e(t)$ is a cosine series which we could find by computing the integrals for the cosine coefficients.

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(nt) dt = \frac{2}{\pi} \int_0^{\pi/2} 2\cos(nt) dt.$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} f(t) dt = \frac{2}{\pi} \int_0^{\pi/2} 2 dt.$$

But let's recognize $\tilde{f}_e(t) = 1 + sq(t + \pi/2)$, where $sq(t)$ is our standard odd period 2π square wave. So,

$$\tilde{f}_e(t) = 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(n(t + \pi/2))}{n} = 1 + \frac{4}{\pi}(\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \frac{\cos(7t)}{7} + ...)$$

We break the problem into pieces and solve: $\dot{x}_n + kx_n = \cos(nt)$.

Characteristic polynomial: $P(r) = r + k$. So, the sinusoidal response formula gives

$$x_{n,p} = \frac{\cos(nt - \phi(n))}{|P(in)|} = \frac{\cos(nt - \phi(n))}{\sqrt{k^2 + n^2}},$$

where $\phi(n) = \text{Arg}(P(in)) = \text{Arg}(k + in) = \tan^{-1}(n/k)$ (in the first quadrant).

Either by applying the above formula with $n = 0$ or directly we find that $x_{0,p}(t) = \frac{1}{k}$.

Using superposition (including the Fourier coefficients), the periodic solution to the DE is

$$\begin{aligned} x_{sp} &= x_{0,p} + \frac{4}{\pi}\left(x_{1,p} - \frac{x_{3,p}}{3} + ...\right) \\ &= \frac{1}{k} + \frac{4}{\pi}\left(\frac{\cos(t - \phi(1))}{\sqrt{k^2+1}} - \frac{\cos(3t - \phi(3))}{3\sqrt{k^2+9}} + \frac{\cos(5t - \phi(5))}{5\sqrt{k^2+25}} - ...\right) \end{aligned}$$

Problem 5.

Match each of the following Fourier series with a graph below. For credit you must give a short explanation of your choice.

(a) $4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nt)+4\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nt)$

Solution: Neither even nor odd function $\Rightarrow$ $\boxed{\text{Graph II.}}$

(b) $\sum_{n=1}^{\infty}\frac{3}{\pi n^2}\cos(n\pi t)$

Solution: Even function, period 2 $\Rightarrow$ $\boxed{\text{Graph V.}}$

(c) $\frac{4}{\pi}\sum_{n\,odd}\frac{\sin(nt)}{n}$.

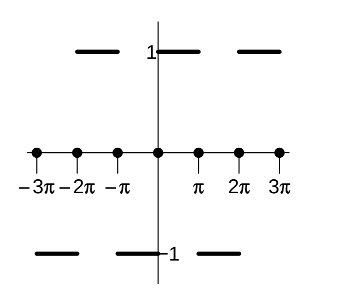
Solution: Odd period 2π square wave has known Fourier series $\Rightarrow$ $\boxed{\text{Graph I.}}$

(d) $\sum_{n=1}^{\infty}\frac{3}{\pi n^2}\cos(nt)$

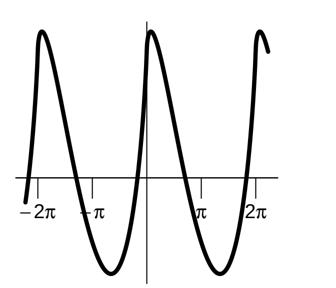
Solution: Even function, period 2π $\Rightarrow$ $\boxed{\text{Graph IV.}}$

(e) $\sum_{n=1}^{\infty}\frac{3}{\pi n^3}\sin(nt)$

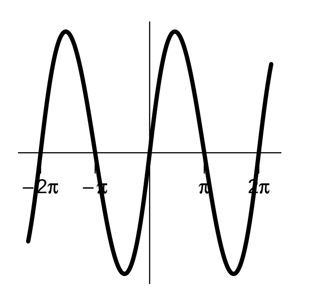
Solution: Odd function, not the square wave $\Rightarrow$ $\boxed{\text{Graph III.}}$



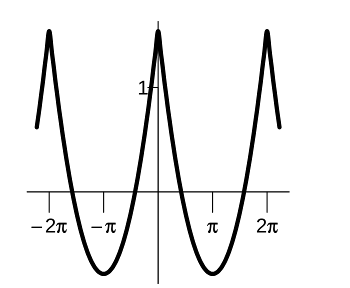
Graph I



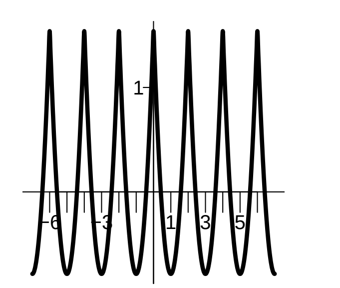
Graph II



Graph III



Graph IV



Graph V

Problem 6.

Use the integral sheet on the first page to compute each of the following.

(a) *The Fourier series of the odd period 2π function that is t^2 on $[0,\pi]$.*

Solution: This is just a sine series. We use the (3′) from integral table:

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2\sin(nt)\,dt = \frac{2}{\pi}\begin{cases}\frac{\pi^2}{n}-\frac{4}{n^3} & \text{for } n \text{ odd}\\ -\frac{\pi^2}{n} & \text{for } n \text{ even}\end{cases} = \begin{cases}\frac{2\pi}{n}-\frac{8}{\pi n^3} & \text{for } n \text{ odd}\\ -\frac{2\pi}{n} & \text{for } n \text{ even}\end{cases}$$

We won't bother writing this out longhand: $f(t) = \sum_{n=1}^{\infty} b_n\sin(nt)$.

(b) *The Fourier series for the period 2π function that is t^2 on $[0,2\pi]$.*

Solution: We have to integrate over one period. In this case it's easiest to integrate over $[0,2\pi]$. Use (3) and (4) from the integral table to get

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2\cos(nt)\,dt = \frac{1}{\pi}\left[\frac{t^2\sin(nt)}{n}+\frac{2t\cos(nt)}{n^2}-\frac{2\sin(nt)}{n^3}\right]_0^{2\pi} = \frac{4}{n^2}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2\sin(nt)\,dt = \frac{1}{\pi}\left[\frac{-t^2\cos(nt)}{n}+\frac{2t\sin(nt)}{n^2}+\frac{2\cos(nt)}{n^3}\right]_0^{2\pi} = -\frac{4\pi}{n}$$

Let's call the function $f(t)$: $\boxed{f(t) = \frac{4\pi^2}{3}+\sum_1^{\infty}\frac{4}{n^2}\cos(nt)-\sum_1^{\infty}\frac{4\pi}{n}\sin(nt).}$

(c) *The cosine series for $f(x)=x^2$ on $[0,\pi/2]$*

Solution: Using the integral table (4), we get

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} x^2\cos(2nx)\,dx = \frac{4}{\pi}\left[\frac{x^2\sin(2nx)}{2n}+\frac{2x\cos(2nx)}{4n^2}-\frac{2\sin(2nx)}{8n^3}\right]_0^{\pi/2} = \frac{(-1)^n}{n^2}$$

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} x^2\,dx = \frac{\pi^2}{6}.$$

$$\boxed{f(x) = \frac{\pi^2}{12}+\sum_1^{\infty}\frac{(-1)^n}{n^2}\cos(2nx) \text{ on } [0,\pi/2]}.$$

(d) *The sine series for $f(x)=x(1-x)$ on [0,1]*

Solution: We use the table entries (1) and (3).

$$\begin{aligned} b_n &= 2\int_0^1 (x-x^2)\sin(n\pi x)\,dx \\ &= 2\left[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2} + \frac{x^2\cos(n\pi x)}{n\pi} - \frac{2x\sin(n\pi x)}{(n\pi)^2} - \frac{2\cos(n\pi x)}{(n\pi)^3}\right]_0^1 \\ &= 2\left[\frac{-(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{-2(-1)^n+2}{(n\pi)^3}\right] \\ &= \begin{cases} \frac{8}{(n\pi)^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \end{aligned}$$

$$\boxed{f(x) = \sum_{n \text{ odd}} \frac{8}{(n\pi)^3}\sin(n\pi x) \text{ on } [0,1].}$$

Problem 7.
Find the Fourier series for each of the following periodic functions (no integrals needed):
(a) $\cos(2t)$, **(b)** $3\cos(2t-\pi/6)$, **(c)** $\cos(t)+2\cos(5t)$, **(d)** $\cos(3t)+\cos(4t)$.

Solution: **(a)** $\cos(2t)$.

(b) $3(\cos(\pi/6)\cos(2t)+\sin(\pi/6)\sin(2t)) = \frac{3\sqrt{3}}{2}\cos(2t)+\frac{3}{2}\sin(2t)$.

(c) itself

(d) itself

Problem 8.
In Problem 7 a-d identify the fundamental frequency and the base period corresponding to that frequency. Using these identify the Fourier coefficients a_n and b_n.

Solution: **(a)** Fundamental frequency $= 2$, base period $= \pi$. $a_1 = 1$ all other coefficients are 0.

(b) Fundamental frequency $= 2$, base period $= \pi$. $a_1 = \sqrt{3}/2$, $b_1 = 1/2$, all other coefficients are 0.

(c) Fundamental frequency $= 1$, base period $= 2\pi$. $a_1 = 1$, $a_5 = 2$, all others are 0.

(d) Fundamental frequency $= 1$ (need all frequencies to be a multiple of this), base period $= 2\pi$.

$a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, all others are 0.

Problem 9.
Compute the Fourier series for the odd, period 2, amplitude 1 square wave. (Do this by computing integrals –not starting with the period 2π square wave.)

Solution: $L = 1$. $f(t) = \begin{cases} -1 & \text{for } -1 \le t < 0 \\ 1 & \text{for } 0 \le t < 1 \end{cases}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right)\,dt = \left(\int_{-1}^{0} -\cos(n\pi t)\,dt + \int_{0}^{1}\cos(n\pi t)\,dt\right) = 0.$$

$$a_0 = \int_{-1}^{1} f(t)\,dt = 0 \text{ (by considering the area under the graph).}$$

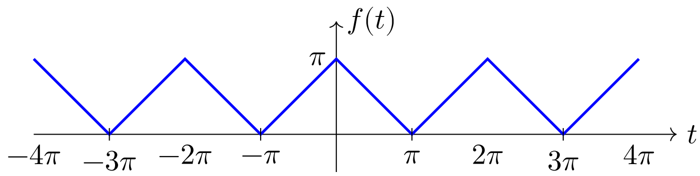
$$b_n = \int_{-1}^{1} f(t)\sin(n\pi t)\,dt = \left(\int_{-1}^{0} -\sin(n\pi t)\,dt + \int_{0}^{1}\sin(n\pi t)\,dt\right)$$

$$= \left[\frac{\cos(n\pi t)}{n\pi}\right]_{-1}^{0} + \left[-\frac{\cos(n\pi t)}{n\pi}\right]_{0}^{1} = \frac{1-(-1)^n}{n\pi} - \frac{(-1)^n - 1}{n\pi} = \begin{cases} 0 & \text{if } n \text{ even} \\ 4/n\pi & \text{if } n \text{ odd} \end{cases}.$$

So, $\boxed{f(t) = \frac{4}{\pi}\sum_{n\,odd}\frac{\sin(n\pi t)}{n}.}$

Problem 10.

Compute the Fourier series for the period 2π triangle wave shown.



Solution: $L = \pi$, $f(t) = \begin{cases} \pi + t & \text{for } -1 < t < 0 \\ \pi - t & \text{for } 0 < t < 1 \end{cases}$.

Since $f(t)$ is even, we know $b_n = 0$ and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\,dt = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos(nt)\,dt$$

$$= \frac{2}{\pi}\left[\frac{\pi\sin(nt)}{n} - \frac{t\sin(nt)}{n} - \frac{\cos(nt)}{n^2}\right]_0^{\pi} = -\frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$= \begin{cases} \frac{4}{\pi n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even.} \end{cases}$$

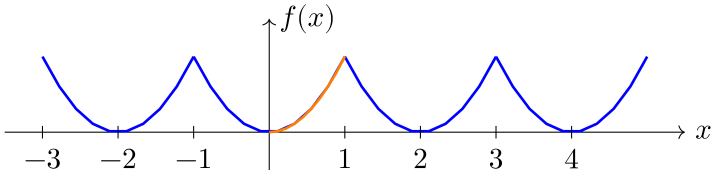
So,

$$f(t) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\cos(nt)}{n^2}.$$

Problem 11.

Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.

Solution: The graph is of the even extension is shown below. $f(x)$ is shown as the orange segment above the interval $[0, 1]$.



We have $L = 1$. The cosine coefficients are computed using the table or by parts.

$$\begin{aligned} a_0 &= 2\int_0^1 f(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}. \\ a_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^1 x^2\cos(n\pi x)\,dx \\ &= 2\left[\frac{x^2\sin(n\pi x)}{n\pi} + \frac{2x\cos(n\pi x)}{(n\pi)^2} - \frac{2\sin(n\pi x)}{(n\pi)^3}\right]_0^1 \\ &= \frac{4(-1)^n}{(n\pi)^2}. \end{aligned}$$

Thus, $f(x) = \dfrac{1}{3} + 4\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2}\cos(n\pi x)$.

Problem 12.
Find the Fourier series for the standard square wave shifted to the left, so that it is an even function, i.e., $sq(t + \pi/2)$.

Solution: Call the standard period 2π, odd, amplitude 1 square wave $sq(t)$. We know that $sq(t) = \dfrac{4}{\pi}\displaystyle\sum_{n\ odd} \frac{\sin(nt)}{n}$

Our function is

$$f(t) = sq(t + \pi/2) = \frac{4}{\pi}\sum_{n\ odd} \frac{\sin(n(t + \pi/2))}{n} = \frac{4}{\pi}\left(\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - ...\right).$$

This last equation follows because

$$\sin(\theta + \pi/2) = \cos(\theta), \quad \sin(\theta + 3\pi/2) = -\cos(\theta), \quad \sin(\theta + 5\pi/2) = \cos(\theta) \ldots.$$

(You can see this either using the trig identity for $\sin(a + b)$ or by thinking about shifting a sine curve to the left by an odd multiple of $\pi/2$.)

Problem 13.
Find the Fourier sine series for $f(t) = 30$ on $[0, \pi]$.

Solution: The sine series is the Fourier series of the odd period 2π extension of f. This is clearly our standard square wave scaled by 30. So,

$$f(t) = \frac{120}{\pi}\sum_{n\ odd} \frac{\sin(nt)}{n}.$$

Problem 14.
Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$.

Solution: We know the Fourier series for $f(t)$, but we'll sketch the computation.

$f(t)$ is even, so $b_n = 0$. We use the evenness to simplify the integral for the cosine coefficients

$$a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \pi, \qquad a_n = \frac{2}{\pi}\int_0^\pi t\cos(nt)\,dt = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \neq 0 \text{ even} \end{cases}$$

So the DE is: $\quad x' + kx = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n\ odd} \frac{\cos(nt)}{n^2}$.

Superposition: We'll solve for each piece first: $\quad x_n' + kx_n = \dfrac{4}{n^2\pi}\cos(nt)$

We use the sinusoidal response formula (SRF). First compute $P(in)$ in polar form.

$$P(in) = k + in = \sqrt{k^2+n^2}\,e^{i\phi(n)}, \text{ where } \boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}(n/k) \text{ in Q1}}.$$

The SRF gives: $\boxed{x_{n,p}(t) = \dfrac{4\cos(nt-\phi(n))}{\pi n^2|P(in)|} = \dfrac{4\cos(nt-\phi(n))}{\pi n^2\sqrt{k^2+n^2}}}$.

Separate calculation for $n = 0$: $\quad x_0' + kx_0 = \pi/2 \quad \Rightarrow x_{0,p}(t) = \pi/2k$.

Superposition:

$$\boxed{x_p(t) = x_{0,p} - \sum_{n\ odd} x_{n,p} = \frac{\pi}{2k} - \frac{4}{\pi}\sum_{n\ odd}\frac{\cos(nt-\phi(n))}{n^2\sqrt{k^2+n^2}}}.$$

Problem 15.
Compute the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\,dt$

(b) $\displaystyle\int_1^5 \delta(t) + 3\delta(t-2) + 6\delta(t-7)\,dt$

(c) $\displaystyle\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\,dt.$

(d) *Make up others.*

(e) *Indefinite integrals: (i)* $\displaystyle\int \delta(t)\,dt$ *(ii)* $\displaystyle\int \delta(t) - \delta(t-3)\,dt.$ *Graph the solutions.*

Solution: (a) Both spikes are inside the interval of integration. So the integral equals 4.

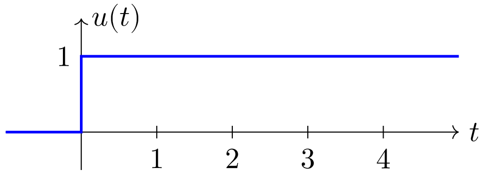
(b) Only the spike at $t = 2$ is inside the interval of integration. So the integral equals 3.

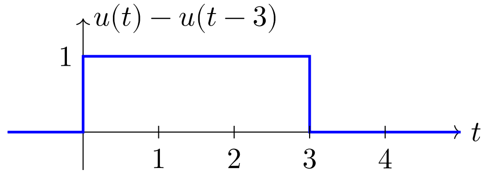
(c) All the spikes are inside the interval of integral so the integral is

$$\cos(0) + \sin(\pi) + \cos(2\pi) = 2.$$

(e) The antiderivative of $\delta(t)$ is $u(t) + C$. So,

(i) $u(t) + C$, (ii) $u(t) - u(t-3) + C$.





Problem 16.

Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC

Solution: Rest IC means that $x(0^-) = 0$.

We work on the intervals between the impulses one at a time.

On $t < 0$: The DE is $x' + 2x = 0$, with $x(0^-) = 0$. The solution is $x(t) = 0$.

On $0 < t < 3$: The DE is $x' + 2x = 0$. The pre-initial condition is $x(0^-) = 0$. The impulse at $t = 0$ produces post-initial conditions $x(0^+) = 1$.

Solving we get $x(t) = ce^{-2t}$. The IC gives $x(0^+) = c = 1$. So, $\boxed{x(t) = e^{-2t}.}$.

The end condition for this interval is $x(3^-) = e^{-6}$.

On $3 < t$: The DE is $x' + 2x = 0$. The pre-initial condition is $x(3^-) = e^{-6}$. The impulse at $t = 3$ produces post-initial conditions $x(3^+) = e^{-6} + 1$.

Solving we get $x(t) = ce^{-2(t-3)}$. The post-initial condition shows $c = 1 + e^{-6}$.

Putting the cases together, we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 < t < 3 \\ (1 + e^{-6})e^{-2(t-3)} & \text{for } t > 3 \end{cases}$$

Problem 17.

(Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.

Solution: Rest IC means $x(0^-) = 0$, $x'(0^-) = 0$.

On $t < 0$: The DE with initial conditions is

$$4\ddot{x} + x = 0; \qquad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

The solution on this interval is $x(t) = 0$.

On $t > 0$: The pre-initial conditions are $x(0^-) = 0, x'(0^-) = 0$. The input $5\delta(t)$ is an impulse which produces post-initial conditions

$$x(0^+) = 0, \quad \dot{x}(0^+) = \dot{x}(0^-) + 5/4 = 5/4.$$

So, we need to solve

$$4\ddot{x} + x = 0; \qquad x(0^+) = 0,\ \dot{x}(0^+) = 5/4.$$

The characteristic roots are $\pm 2i$. So the general solution to the DE is

$$x(t) = c_1 \cos(t/2) + c_2 \sin(t/2).$$

We need to find c_1 and c_2 to match the post-initial conditions. The algebra yields $c_1 = 0$, $c_2 = 5/2$.

Putting the cases together, the complete solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{5}{2}\sin(t/2) & \text{for } t > 0. \end{cases}$$

If we added more impulses to the input, you would work one interval at a time with each impulse causing a jump in momentum.

Problem 18.

Solve $x' + 3x = \delta(t) + e^{2t} + \delta(t-4)$ with $x(0^-) = 0$.

Solution: We use superposition to find a solution.

Piece 1: $x_1' + 3x_1 = \delta(t)$, $x(0^-) = 0$.

We know $x_1(t) = 0$ for $t < 0$. The impulse causes a jump of 1 in x at 0, so the post initial condition is $x(0^+) = 1$. For $t > 0$ the DE is

$$x_1' + 3x_1 = 0, \qquad x(0^+) = 1.$$

This is easy to solve:

$$x_1(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-3t} & \text{for } t > 0. \end{cases}$$

Piece 2: $x_2' + 3x_2 = e^{2t}$, $x(0^-) = 0$.

Using the ERF and the general homogeneous solution: $x_2(t) = \dfrac{e^{2t}}{5} + Ce^{-3t}$.

The IC $x(0) = 0$ implies $C = -1/5$.

So, $x_2(t) = \dfrac{e^{2t}}{5} - \dfrac{e^{-3t}}{5}$.

Piece 3: $x_3' + 3x_3 = \delta(t-4)$, $x(0^-) = 0$.

Similar to Piece 1, we'll have $x_3(t) = 0$ for $t < 4$. So, $x(4^-) = 0$. Now, this is identical to Piece 1, except shifted to $t = 4$

$$x_3(t) = \begin{cases} 0 & \text{for } t < 4 \\ e^{-3(t-4)} & \text{for } t > 4. \end{cases}$$

Putting the pieces together, the solution to the problem is

$$x(t) = x_1(t) + x_2(t) + x_3(t) = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5} + \begin{cases} 0 & \text{for } t < 0 \\ e^{-3t} & \text{for } 0 < t < 4 \\ e^{-3t} + e^{-3(t-4)} & \text{for } t > 4. \end{cases}$$

Note: You should verify, that, since each piece satisfies the rest initial condition, so does their sum. That is, we have homogeneous IC. If the IC was inhomogeneous, we would want just one of the pieces to satisfy the inhomogeneous IC and the others to satisfy the homogeneous IC.

Problem 19.
Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.

Solution: Rest IC means $x(0^-) = 0$, $x'(0^-) = 0$.

<u>On $t < 0$</u>: The differential equation with initial conditions is

$$2x'' + 8x' + 6x = 0; \qquad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

The solution to this is $x(t) = 0$.

<u>On $t > 0$</u>: The impulse at $t = 0$ causes a jump in x'. That is, we have post-initial conditions $x(0^+) = 0$, $x'(0^+) = x'(0^-) + 1/2 = 1/2$.

So the differential equation with initial conditions is

$$2x'' + 8x' + 6x = 0; \qquad x(0^+) = 0,\ \dot{x}(0^+) = 1/2.$$

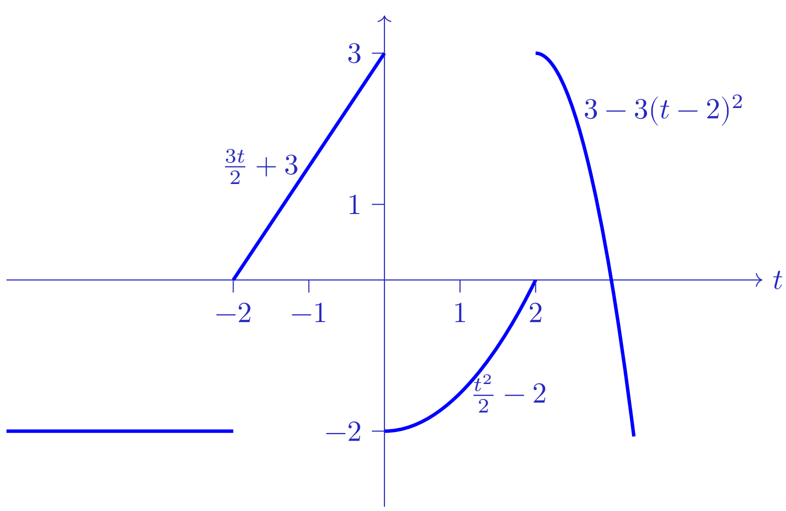
The characteristic roots are -1, -3. So the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

We find c_1 and c_2 to match the post-initial conditions: $c_1 = 0$, $c_2 = 1/2$. The complete solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{-t}}{4} - \frac{e^{-3t}}{4} & \text{for } 0 < t \end{cases}$$

Problem 20.
The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.



Solution: The formula for the function is

$$f(t) = \begin{cases} -2 & \text{for } t < -2 \\ \frac{3t}{2} + 3 & \text{for } -2 < t < 0 \\ \frac{t^2}{2} - 2 & \text{for } 0 < t < 2 \\ 3 - 3(t-2)^2 & \text{for } 2 < t. \end{cases}$$

We have to take the regular derivative and add delta functions at the jump discontinuities.

$$f'(t) = \underbrace{2\delta(t+2) - 5\delta(t) + 3\delta(t-3)}_{\text{singular part}} + \underbrace{\begin{cases} 0 & \text{for } t < -2 \\ \frac{3}{2} & \text{for } -2 < t < 0 \\ t & \text{for } 0 < t < 2 \\ -6(t-2) & \text{for } 2 < t. \end{cases}}_{\text{regular part}}$$

End of practice quiz solutions