ES.1803 Practice Questions – Quiz 5, Spring 2024
Covers Topics 20-23

On the quiz you will be given the following table:

Integrals (for n a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$

Some Fourier series:

1. Period 2π square wave $\mathrm{sq}(t)$: *You should know this for the quiz.*
2. Period 2 triangle wave $\mathrm{tri2}(t)$:

   Over one period, $-1 \le t \le 1$, $\mathrm{tri2}(t) = |t|$.

$$\begin{aligned}\mathrm{tri2}(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right)\\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}.\end{aligned}$$

This will probably take longer than 1 hour; the actual quiz will be considerably shorter. Not every possible question is covered. For more questions, you can also look at the in-class problems, problem section worksheets and psets.

Be wise in carrying out computations. If you get a problem to a point where you are sure –*emphasize sure*– you can carry out the computation, then stop and go on to the next problem. At the end you can check the solutions to see that you had correctly set up the computation.

Problem 1. (Step and delta)
(a) Compute the following integrals.

(i) $\displaystyle\int_{-\infty}^{\infty} e^{\lambda t}(\delta(t) + 2\delta(t-1) + 3\delta(t-2))\,dt$

(ii) $f(t) = \displaystyle\int_{0^-}^{t} 2\delta(\tau+1) + 3\delta(\tau) + 4\delta(\tau-1)\,d\tau$. Assume $t > 0$. What is $f(3)$?

(iii) $f(t) = \displaystyle\int e^{\lambda t}(\delta(t) + 2\delta(t-1) + 3\delta(t-2))\,dt$

(b) Solve the following initial value problems.

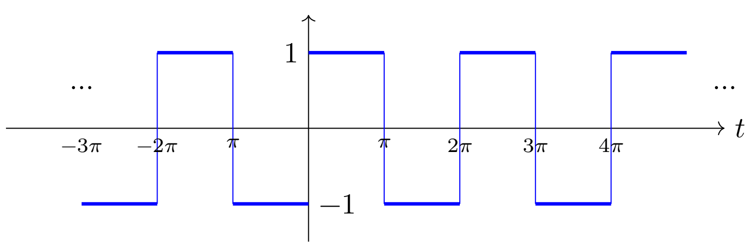
(i) $2\ddot{x} + 7\dot{x} + 3x = \delta(t)$ with rest initial conditions.

(ii) $2\ddot{x} + 7\dot{x} + 3x = \delta(t) + e^{3t}$ with $x(0^-) = 0$, $\dot{x}(0^-) = 0$.

(c) Compute the Fourier series for the period 1 impulse train

$$f(t) = \ldots + \delta(t+2) + \delta(t+1) + \delta(t) + \delta(t-1) + \ldots$$

Problem 2. Derivative of a square wave
The graph below is of the function $\mathrm{sq}(t)$ (standard square wave). Compute and graph its generalized derivative.



Graph of $\mathrm{sq}(t)$ = square wave

Problem 3.
Let $f(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ 0 & 1 \le x < 2 \end{cases}$

(a) Sketch the following periodic extensions of f over three or more full periods:

(i) even period 4 extension (ii) odd period 4 extension (iii) periodic extension with period 2.

For (ii) and (iii), also sketch in the 'extra' points to which the Fourier series expansion will converge (*without* computing the Fourier series).

(b) Compute the Fourier cosine series of f.

(c) Write out (with decimal numbers) the first 4 terms of the series in Part (b).

(d) Compute the steady periodic solution to the DE $x''(t) + 2.5\,x(t) = \tilde{f}_e(t)$ where $\tilde{f}_e$ is the even periodic extension of f.

Problem 4.

Let $\tilde{f}_e(t)$ be the even period 2π extension to the function $f(t) = \begin{cases} 2 & \text{if } 0 < t < \pi/2 \\ 0 & \text{if } \pi/2 < t < \pi. \end{cases}$

Solve $\dot{x} + kx = \tilde{f}_e(t)$ for the periodic solution in Fourier series form,

Problem 5.

Match each of the following Fourier series with a graph below. For credit you must give a *short* explanation of your choice.

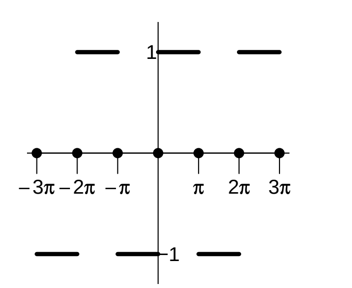
(a) $4\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nt) + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nt)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\pi\, n^2}\cos(n\pi t)$

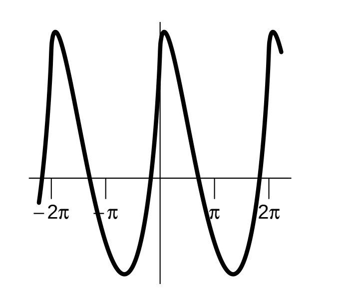
(c) $\displaystyle\frac{4}{\pi}\sum_{n\ odd} \frac{\sin(nt)}{n}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\pi\, n^2}\cos(nt)$

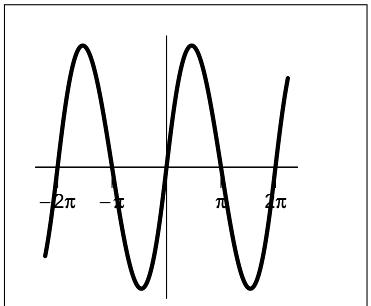
(e) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\pi\, n^3}\sin(nt)$



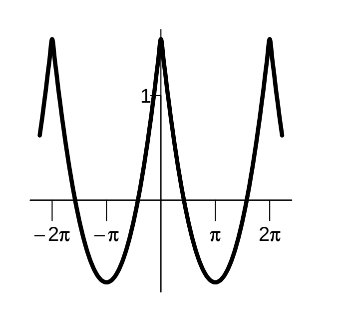
Graph I



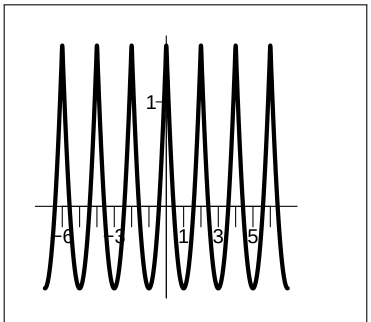
Graph II



Graph III



Graph IV



Graph V

Problem 6.
Use the integral sheet on the first page to compute each of the following.

(a) The Fourier series of the odd period 2π function that is t^2 on $[0, \pi]$.

(b) The Fourier series for the period 2π function that is t^2 on $[0, 2\pi]$.

(c) The cosine series for $f(x) = x^2$ on $[0, \pi/2]$

(d) The sine series for $f(x) = x(1 - x)$ on [0,1]

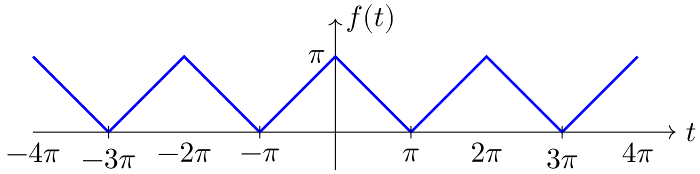
Problem 7.
Find the Fourier series for each of the following periodic functions (no integrals needed):

(a) $\cos(2t)$, **(b)** $3\cos(2t - \pi/6)$, **(c)** $\cos(t) + 2\cos(5t)$, **(d)** $\cos(3t) + \cos(4t)$.

Problem 8.
In Problem 7 a-d identify the fundamental frequency and the base period corresponding to that frequency. Using these identify the Fourier coefficients a_n and b_n.

Problem 9.
Compute the Fourier series for the odd, period 2, amplitude 1 square wave. (Do this by computing integrals –not starting with the period 2π square wave.)

Problem 10.
Compute the Fourier series for the period 2π triangle wave shown.



Problem 11.
Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.

Problem 12.
Find the Fourier series for the standard square wave shifted to the left, so that it is an even function, i.e., $sq(t + \pi/2)$.

Problem 13.
Find the Fourier sine series for $f(t) = 30$ on $[0, \pi]$.

Problem 14.
Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$.

Problem 15.
Compute the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\, dt$

(b) $\displaystyle\int_{1}^{5} \delta(t) + 3\delta(t-2) + 6\delta(t-7)\, dt$

(c) $\displaystyle\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\, dt.$

(d) Make up others.

(e) Indefinite integrals: (i) $\displaystyle\int \delta(t)\, dt$ (ii) $\displaystyle\int \delta(t) - \delta(t-3)\, dt.$ Graph the solutions.

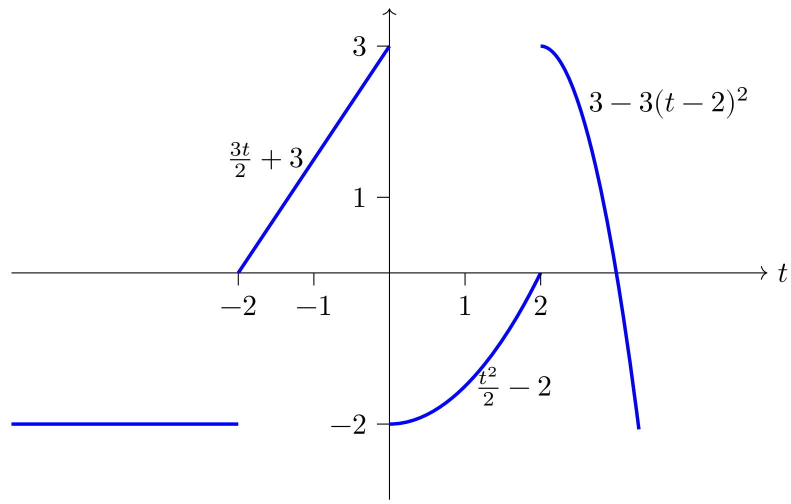
Problem 16.
Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC

Problem 17.
(Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.

Problem 18.
Solve $x' + 3x = \delta(t) + e^{2t} + \delta(t-4)$ with $x(0^-) = 0$.

Problem 19.
Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.

Problem 20.
The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.



End of practice quiz