ES.1803 Practice Solutions – Quiz 6, Spring 2024

Integrals (for n a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$

Some Fourier series:

1. Period 2π square wave $\text{sq}(t)$: *You should know this for the quiz.*
2. Period 2 triangle wave $\text{tri2}(t)$:

   Over one period, $-1 \le t \le 1$, $\text{tri2}(t) = |t|$.

$$\begin{aligned}\text{tri2}(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right) \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}.\end{aligned}$$

Problem 1.
Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$.

Solution: We know the Fourier series for $f(t)$, but we'll sketch the computation.

$f(t)$ is even, so $b_n = 0$. We use the evenness to simplify the integral for the cosine coefficients

$$a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \pi, \qquad a_n = \frac{2}{\pi}\int_0^\pi t\cos(nt)\,dt = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \neq 0 \text{ even} \end{cases}$$

So the DE is: $\quad x' + kx = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n\ odd} \frac{\cos(nt)}{n^2}.$

Superposition: We'll solve for each piece first: $\quad x_n' + kx_n = \cos(nt)$

In preparation for using the sinusoidal response formula (SRF), we first compute $P(in)$ in polar form.

$$P(in) = k + in = \sqrt{k^2 + n^2}\, e^{i\phi(n)}, \text{ where } \boxed{\phi(n) = \operatorname{Arg}(P(in)) = \tan^{-1} n/k \text{ in Q1}}.$$

The SRF gives: $\boxed{x_{n,p}(t) = \dfrac{\cos(nt - \phi(n))}{|P(in)|} = \dfrac{\cos(nt - \phi(n))}{\sqrt{k^2 + n^2}}.}$

Separate calculation for $n = 0$: $\quad x_0' + kx_0 = \pi/2 \Rightarrow x_{0,p} = \pi/2k.$

Superposition:

$$\boxed{x_p(t) = x_{0,p} - \frac{4}{\pi}\sum_{n\ odd} \frac{x_{n,p}(t)}{n^2} = \frac{\pi}{2k} - \frac{4}{\pi}\sum_{n\ odd} \frac{\cos(nt - \phi(n))}{n^2\sqrt{k^2 + n^2}}}.$$

Problem 2.
Solve $x'' + 2x' + 9x = g(t)$ where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$.

Solution: This is just tri2(x) given in the integral table above:

$$g(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\ odd} \frac{\cos(n\pi t)}{n^2}.$$

(Or, if you remember, the Fourier series for our standard period 2π triangle wave tri(t), you can use $g(t) = \text{tri}(\pi t)/\pi$. Or you can just compute the integrals for the coefficients.)

Use the SRF to solve for each piece:

$$x_n'' + 2x_n' + 9x_n = \cos(n\pi t).$$

First we find $P(in)$ in polar form: $P(i\pi n) = 9 - (\pi n)^2 + 2i\pi n = \sqrt{(9 - \pi^2 n^2)^2 + 4\pi^2 n^2}\, e^{i\phi(n)}$, where $\boxed{\phi(n) = \operatorname{Arg}(P(in)) = \tan^{-1}(2n\pi/(9 - \pi^2 n^2)) \text{ in Q1 or Q2.}}$

So, $x_{n,p}(t) = \dfrac{\cos(n\pi t - \phi(n))}{\sqrt{(9 - \pi^2 n^2)^2 + 4\pi^2 n^2}}.$

Separate calculation for $n=0$: $x_0'' + 2x_0' + 9x_0 = \dfrac{1}{2} \Rightarrow x_{0,p} = 1/18$.

Superposition:
$$\boxed{x_p(t) = \frac{1}{18} - \frac{4}{\pi^2} \sum_{n\,odd} \frac{\cos(n\pi t - \phi(n))}{n^2\sqrt{(9-\pi^2 n^2)^2 + 4\pi^2 n^2}}}.$$

(Don't forget, you should show the dependence on n by writing $\phi(n)$.)

Problem 3.

Solve $x'' + 4x = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$. Look out for resonance.

Solution: Solve this in pieces: $x_n'' + 4x_n = \cos(nt)$.

The characteristic polynomial is $P(r) = r^2 + 4$. So, $P(in) = 4 - n^2$ and

$$|P(in)| = |4 - n^2|, \qquad \phi(n) = \text{Arg}(P(in)) = \begin{cases} 0 & \text{for } n < 2 \\ \pi & \text{for } n > 2 \end{cases}$$

We use the SRF for the cases $n \neq 2$:

$$x_{n,p}(t) = \frac{\cos(nt - \phi(n))}{|P(in)|} = \frac{\cos(nt - \phi(n))}{|4 - n^2|}.$$

For the case $n = 2$, we need the extended SRF: $P'(2i) = 4i$. So, $|P'(2i)| = 4$ and $\text{Arg}(P'(2i)) = \pi/2$. Thus,

$$x_{2,p}(t) = \frac{t\cos(2t - \pi/2)}{4}.$$

By superposition

$$\boxed{x_p(t) = \sum_{n=1}^{\infty} \frac{x_{n,p}(t)}{n^2} = \frac{\cos(t)}{3} + \frac{t\sin(2t)}{2^2 \cdot 4} + \frac{\cos(3t - \pi)}{3^2 \cdot 5} + \frac{\cos(4t - \pi)}{4^2 \cdot 12} + \frac{\cos(5t - \pi)}{5^2 \cdot 21} + \ldots}.$$

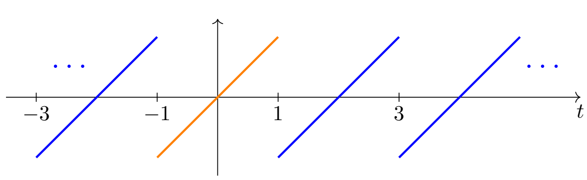
Since $\cos(nt - \pi) = -\cos(nt)$, we can also write

$$\boxed{x_p(t) = \frac{\cos(t)}{3} + \frac{t\sin(2t)}{16} - \frac{\cos(3t)}{45} - \frac{\cos(4t)}{192} - \frac{\cos(5t)}{525} - \ldots}.$$

Problem 4.

(a) *The function $f(t)$ is periodic with period 2. On the interval $-1 \leq t < 1$ we have $f(t) = t$. Find the Fourier series for $f(t)$.*

Solution: Here is the graph of $f(t)$:



We have $L = 1$ and $f(t)$ is odd $\Rightarrow$ $f(t) = \sum b_n \sin(n\pi t)$, where

$$b_n = 2\int_0^1 f(t)\sin(n\pi t)\,dt \;=\; 2\int_0^1 t\sin(n\pi t)\,dt = 2\left[-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{(n\pi t)^2}\right]_0^1 = (-1)^{(n+1)}\frac{2}{n\pi}.$$

So, $\boxed{f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{\sin(n\pi t)}{n}}$.

(b) (10)*Find a periodic solution to $x'' + 36x = f(t)$.*

Solution: We will use superposition, so first we solve individual equations

$$x_n'' + 36x_n = \sin(n\pi t).$$

We have $P(in\pi) = 36 - n^2\pi^2$. So, $|P(in\pi)| = |36 - n^2\pi^2|$, $\boxed{\phi(n) = \text{Arg}(P(in\pi)) = \begin{cases} 0 & \text{if } n = 1 \\ \pi & \text{if } n > 1 \end{cases}}$.

Thus, $x_{n,p}(t) = \frac{\sin(n\pi t - \phi(n))}{|36 - n^2\pi^2|}$.

Now by superposition: $\boxed{x_p(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{\sin(n\pi t - \phi(n))}{n|36 - n^2\pi^2|}}$.

(c) (5)*Which frequency in the Fourier series for $f(t)$ is closest to resonance for the system in Part (b).*

Solution: The natural frequency is $\sqrt{36} = 6$. The frequency of the n^{th} term is $n\pi$ $\Rightarrow$ the term with $\boxed{n = 2}$ is closest to resonance

Problem 5. *(Heat equation with boundary and initial conditions)*

For $0 \le x \le \pi$ and $t > 0$ we have

PDE: $u_t = u_{xx}$

BC: $u(0,t) = 0$ *and* $u(\pi,t) = 0$ *for all $t > 0$.*

IC: $u(x,0) = f(x)$ *for $0 \le x \le \pi$.*

Showing all the steps clearly, use the separation of variables method to get the general solution $u(x,t)$ which satisfies both the PDE and BC.

Then give formulas for the Fourier coefficients of the solution which also satisfies the IC . These formulas will have to be given in terms of f.

Solution: We break the method into steps.

Step 1. Find separated solutions to the PDE, i.e., guess $u(x,t) = X(x)T(t)$.

Plug into PDE: $XT' = X''T$

A little algebra: $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \text{constant} = -\lambda$. (It equals a constant because x and t are independent variables.)

More algebra gives two ODEs: $X'' + \lambda X = 0$, $T' + \lambda T = 0$.

We have three cases:

Case (i) $\lambda > 0$: $\quad X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x), \quad T(t) = e^{-\lambda t}$.

Case (ii) $\lambda = 0$: $\quad X(x) = a + bx, \quad T(t) = c$.

Case (iii) $\lambda < 0$: This case never produces nontrivial solutions satisfying the BC, so we ignore it.

Step 2. (Modal solutions) Find the separated solutions which also satisfy the BC.

For separated solutions, the BC are

$$X(0) = 0, \quad X(\pi) = 0.$$

Case (i) BC: $X(0) = a = 0$ and $X(\pi) = a\cos(\sqrt{\lambda}\pi) + b\sin(\sqrt{\lambda}\pi)$.

The nontrivial solutions have $a = 0$, b arbitrary and $\sqrt{\lambda} = n$, for $n = 1, 2, 3, ...$ Thus, the separated solutions satisfying the PDF and BC are

$$u_n(x,t) = b_n \sin(nx) c_n e^{-n^2 t}.$$

Case (ii) BC: $X(0) = a = 0$ and $X(\pi) = a + b\pi = 0$.

Solving, we get $a = 0$ and $b = 0$, i.e., this case only produces trivial solutions.

Case (iii) Ignore.

Step 3. Use superposition to get the general solution satisfying both the PDF and the BC.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}}$$

Step 4. Use the initial conditions to compute the coefficients.

$u(x,0) = \sum b_n \sin(nx) = f(x)$ on the interval $[0,\pi]$. So b_n are the Fourier sine coefficients for $f(x)$. Since $f(x)$ is not specified, the best we can do is:

$$\boxed{b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.}$$

Problem 6.

(a) *Write down the wave equation with IC's and BC's for the string of length 1, with clamped ends, wave speed 2, initially at equilibrium, struck at time 0. Then derive the Fourier series solution using separation of variables.*

Solution: PDE: $y_{tt} = 4y_{xx}$

BC: $y(0,t) = y(1,t) = 0$

IC: $y(x,0) = 0$ (initially at equilibrium)
$y_t(x,0) = f(x)$ (initial velocity right after being struck)

We solve using the Fourier method of separation of variables.

Step 1. Find separated solutions to the PDE: $y(x,t) = X(x)T(t)$

Substituting into the PDE:

$$XT'' = 4X''T \quad \Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{4T(t)} = \text{ constant } = -\lambda.$$

(Since x and t are independent, a function of x = function of t implies both must be constant.)

Thus we have two ODES: $X'' + \lambda X = 0$, $T'' + 4\lambda T = 0$.

As always, we have cases.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$, $T(t) = c\cos(2\sqrt{\lambda}\,t) + d\sin(2\sqrt{\lambda}\,t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c + dt$

Case (iii) $\lambda < 0$: Ignore –only gives trivial modal solutions.

Step 2. (Modal solutions) Find the separated solutions which also satisfy the BC.

For separated solutions, the BC are

$$X(0) = 0, \quad X(1) = 0.$$

Case (i) BC: $X(0) = a = 0$, $X(1) = a\sin(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

The nontrivial solutions have $a = 0$, b arbitrary and $\sin(\sqrt{\lambda}) = 0$. This implies $\sqrt{\lambda} = n\pi$, where $n = 1, 2, ...$

Index solutions by n: $X_n(x) = \sin(n\pi x)$; $T_n(t) = c_n\cos(2n\pi t) + d_n\sin(2n\pi t)$; So,

$$y_n(x,t) = X_n(x)T_n(t) = \sin(n\pi x)(c_n\cos(2n\pi t) + d_n\sin(2n\pi t)).$$

Case (ii) BC: $X(0) = a = 0$, $X(1) = a + b = 0$.

Thus, $a = 0$, $b = 0$, i.e., there are only trivial solutions in this case.

Case (iii) We know this case only produces trivial solutions, so we skip it.

Step 3. Use superposition to give the general solution.

$$y(x,t) = \sum y_n(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)(c_n\cos(2n\pi t) + d_n\sin(2n\pi t)).$$

Step 4. Use the initial conditions to compute the coefficients.

$y(x,0) = \sum_{n=1}^{\infty} c_n\sin(n\pi x) = 0$. So $c_n = 0$ for all n.

$y_t(x,0) = \sum_{n=1}^{\infty} d_n 2n\pi\sin(n\pi x) = f(x)$. So $d_n 2n\pi$ are the Fourier sine coefficients of the function $f(x)$ on $[0,1]$. That is,

$$d_n 2n\pi = 2\int_0^1 f(x)\sin(n\pi x)\,dx \quad \text{or} \quad \boxed{d_n = \frac{1}{n\pi}\int_0^1 f(x)\sin(n\pi x)\,dx}.$$

With d_n as just defined, the Fourier solution to the problem is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} d_n \sin(n\pi x)\sin(2n\pi t)}.$$

(b) *Give the explicit solution to the equation of Part (a) when the initial velocity is given by $f(x) = x$ on $0 < x < 1$ (as if that were possible!).*

Solution: From the solution to Part (a) and the integral table we have

$$\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi} \qquad \Rightarrow d_n = \frac{1}{n\pi}\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n^2\pi^2}.$$

Thus, $\boxed{y(x,t) = \sum_{n=1}^{\infty} d_n \sin(n\pi x)\sin(2n\pi t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2}\sin(n\pi x)\sin(2n\pi t)}$.

Problem 7.

Consider the following partial differential equation with boundary and initial conditions:

PDE: $u_t(x,t) + u(x,t) = u_{xx}(x,t)$; *defined for* $0 < x < 1$.

BC: $u(0,t) = 0$, $u(1,t) = 0$.

IC: $u(x,0) = f(x)$.

(a) *The separation of variables technique looks for solutions to the PDE of the form $u(x,t) = X(x)T(t)$. Give the ordinary DEs satisfied by X and T.*

You do not have to solve these DEs.

Solution: Separated solution: $u(x,t) = X(x)T(t)$.

Substitution: $XT' + XT = X''T \Rightarrow \dfrac{T'+T}{T} = \dfrac{X''}{X} = -\lambda$ for some constant λ.

(We must have a constant because we have a function of t = a function of x.)

$\Rightarrow \boxed{X'' + \lambda X = 0 \quad \text{and} \quad T' + (1+\lambda)T = 0.}$

(b) *To make your life easier, we'll tell you that, in the usual notation, the only separated solutions satisfying the boundary conditions have $\lambda > 0$ and are of the form*

$$X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \;\; \textit{and} \;\; T(t) = e^{-(1+\lambda)t}.$$

Of course, not all $\lambda > 0$ work. Find all the separated solutions to the PDE that satisfy the boundary conditions. Then give the general solution to the PDE with BC.

Solution: For separated solutions the BC are $X(0) = 0$, $X(1) = 0$.

BC: $X(0) = a = 0$, $X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

With $a = 0$, the second equation becomes $b\sin(\sqrt{\lambda}) = 0$. If $b = 0$, we have a trivial solution. So, the nontrivial solutions have $\sin(\sqrt{\lambda}) = 0$, $\Rightarrow \sqrt{\lambda} = n\pi$ for some positive integer n.

Modal solutions: $\boxed{u_n(x,t) = b_n\sin(n\pi x)\,e^{-(1+n^2\pi^2)t} \text{ for } n = 1,\, 2,\, 3,\, \ldots}$.

General solution by superposition: $\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-(1+n^2\pi^2)t}}$.

(c) *Give the Fourier solution to PDE with BC and IC. Be sure to write down the integral formula for any coefficients used. (Since f is not specified you cannot compute the integrals.)*

Solution: IC: $u(x,0) = \sum b_n \sin(n\pi x) = f(x)$. So, b_n = Fourier sine coefficient of f:

$$\boxed{b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx.}$$

(d) *We can add input to the PDE: $u_{xx} = u_t + u + xe^{-t}$.*
A particular solution to this PDE also satisfying the BC of Part (a) is: $u_p(x,t) = \left(\frac{x^3}{6} - \frac{x}{6}\right)e^{-t}$.
What are all the solutions to this PDE which also satisfy the BC?

Solution: Using linearity, the general solution is the particular solution + the general homogeneous solution found in Part (c): $\boxed{u(x,t) = \left(\frac{x^3}{6} - \frac{x}{6}\right)e^{-t} + \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-(1+n^2\pi^2)t}}$.

Note well: When using superposition, you need to make sure that it applies to both the PDE and the BC.

Problem 8.
Solve the wave equation with boundary and initial conditions.

PDE: $y_{tt} = y_{xx}$ *for* $0 \le x \le 1$, $t > 0$

BC: $y(0,t) = 0$, $y(1,t) = 0$

IC: $y(x,0) = 0$, $y_t(x,0) = 30$.

Solution: Step 1. Find separated solutions to the PDE: $y(x,t) = X(x)T(t)$.

Substituting into the PDE:

$$XT'' = X''T, \qquad \text{a little algebra gives} \qquad \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \text{constant} \; = -\lambda$$

Some algebra gives two ordinary differential equations

$$X'' + \lambda X = 0 \qquad T'' + \lambda T = 0.$$

For X the characteristic roots are $r = \pm\sqrt{-\lambda}$. There are 3 cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$.

For this λ we have $T(t) = c\cos(\sqrt{\lambda}\,t) + d\sin(\sqrt{\lambda}\,t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$.

For this λ we have $T(t) = c + dt$.

Case (iii) $\lambda < 0$: Never gives nontrivial modal solutions.

Step 2. (Modal solutions) Find the separated solutions which also satisfy the BC.

For separated solutions, the BC are

$$X(0) = 0, \; X(1) = 0.$$

Case (i) BC: $X(0) = a = 0, \quad X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0.$

With $a = 0$, the second condition is $b\sin(\sqrt{\lambda}) = 0$. If $b = 0$, the solution is trivial. So, we need $\sin(\sqrt{\lambda}) = 0$, i.e., $\sqrt{\lambda} = n\pi$ for any integer n.

Index solutions by n: $X_n(x) = b_n \sin(n\pi x)$, $T_n(t) = c\cos(n\pi t) + d\sin(n\pi t)$. So, we have modal solutions

$$y_n(x,t) = \sin(n\pi x)(c_n \cos(n\pi t) + d_n \sin(n\pi t)) \qquad \text{for } n = 1,\, 2, \ldots$$

Case (ii) BC: $X(0) = a = 0, \quad X(1) = a + b = 0.$

Thus, $a = 0$, $b = 0$, i.e., there are only trivial solutions in this case.

Case (iii) $\lambda < 0$: We can always ignore this case.

Step 3. Use superposition to get the general solution.

Using superposition we get that the general solution to the PDE + BC is

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)(c_n \cos(n\pi t) + d_n \sin(n\pi t))$$

Step 4. Use the initial conditions to determine the coefficients.

IC $y(x,0) = 0$: $y(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) = 0$. This is a Fourier sine series for 0, i.e., all the coefficients $c_n = 0$.

IC $y_t(x,t) = 30$: $y_t(x,0) = \sum_{n=1}^{\infty} n\pi d_n \sin(n\pi x) = 30$. This is a Fourier sine series for 30 on [0,1]. We recognize this as the Fourier series for the odd period 2 square wave.

$$\sum_{n=1}^{\infty} n\pi d_n \sin(n\pi x) = 30 = \frac{120}{\pi} \sum_{n\ odd} \frac{\sin(n\pi x)}{n}.$$

So, $n\pi d_n = \begin{cases} \frac{120}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$. We have $\boxed{d_n = \begin{cases} \frac{120}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}}$

Our solution is

$$y(x,t) = \sum_{n=1}^{\infty} d_n \sin(n\pi x)\sin(n\pi t) = \frac{120}{\pi^2} \sum_{n \text{ odd}} \frac{\sin(n\pi x)\sin(n\pi t)}{n^2}.$$

Problem 9.

Solve the heat equation with insulated ends.

(Here's a problem that gives a cosine series so the $\lambda = 0$ case is important.)

PDE: $u_t = 3u_{xx}$ *for* $0 \le x \le 1,\ t > 0$

BC: $u_x(0,t) = 0,\ u_x(1,t) = 0$

IC: $u(x,0) = x$.

Solution: We'll do this with fewer words than in previous problems.

Step 1. Separated solutions: Try $u(x,t) = X(x)T(t)$.

Substitution gives

$$XT' = 3X''T \quad \Rightarrow \quad \frac{X(x)''}{X(x)} = \frac{T(t)'}{3T(t)} = \text{constant} = -\lambda \quad \Rightarrow X'' + \lambda X = 0,\ T' + 3\lambda T = 0.$$

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$, $T(t) = ce^{-3\lambda t}$

So, $u(x,t) = (a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x))\,ce^{-3\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c$. So, $u(x,t) = (a + bx)c$,

Case (iii) $\lambda < 0$: Ignore, this case never produces nontrivial modal solutions.

Step 2. (Modal solutions) Find the separated solutions which also satisfy the BC.

For separated solutions, the BC are $X'(0) = 0$, $X'(1) = 0$.

Case (i) BC: $X'(0) = \sqrt{\lambda}\,b = 0$, $X'(1) = -a\sin(\sqrt{\lambda}) + b\cos(\sqrt{\lambda}) = 0$.

With $b = 0$, the second condition is $-a\sin(\sqrt{\lambda}) = 0$. For nontrivial solutions, we need $\sin(\sqrt{\lambda}) = 0$. That is, $\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$.

We have found modal solutions $\boxed{u_n(x,t) = a_n\cos(n\pi x)e^{-3(n\pi)^2 t} \quad \text{for } n = 1, 2, \ldots}$

(We combined a and c into one constant and added the index n.)

Case (ii) BC: $X'(0) = b = 0$, $X'(1) = b = 0$.

So, $b = 0$ and a is arbitrary, i.e., $X(x) = a$.

We have found one more modal solution. Let's call it $\boxed{u_0 = a_0/2}$.

Case (iii) $\lambda < 0$: Never produces nontrivial solutions.

Step 3. Superposition gives the general solution to PDE + BC.

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x)e^{-3(n\pi)^2 t}$$

Step 4. IC: $u(x,0) = a_0/2 + \sum_n a_n\,\cos(n\pi x) = x$.

This is the cosine series for x. The cosine series for x is the same as the Fourier series for the triangle wave, $\text{tri2}(x)$ in the table.

$$\text{tri2}(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi x)}{n^2}, \qquad \text{i.e., } a_0 = 1,\ a_n = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Thus,

$$\boxed{u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\,\text{odd}} \frac{\cos(n\pi x)\, e^{-3(n\pi)^2 t}}{n^2}}.$$

End of practice quiz solutions