ES.1803 Practice Questions – Quiz 6, Spring 2024
Covers Topics 24-26

On the quiz you will be given the following table:

Integrals (for n a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$

Some Fourier series:

1. Period 2π square wave sq(t): *You should know this for the quiz.*
2. Period 2 triangle wave tri2(t):

   Over one period, $-1 \le t \le 1$, $\text{tri2}(t) = |t|$.

$$\begin{aligned}\text{tri2}(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right) \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}.\end{aligned}$$

This will probably take longer than 1 hour; the actual quiz will be considerably shorter. Not every possible question is covered. For more questions, you can also look at the in-class problems, problem section worksheets and psets.

Be wise in carrying out computations. If you get a problem to a point where you are sure –*emphasize sure*– you can carry out the computation, then stop and go on to the next problem. At the end you can check the solutions to see that you had correctly set up the computation.

Problem 1.
Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$.

Problem 2.
Solve $x'' + 2x' + 9x = g(t)$ where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$.

Problem 3.
Solve $x'' + 4x = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$. Look out for resonance.

Problem 4.
(a) The function $f(t)$ is periodic with period 2. On the interval $-1 \le t < 1$ we have $f(t) = t$. Find the Fourier series for $f(t)$.

(b) (10) Find a periodic solution to $x'' + 36x = f(t)$.

(c) (5) Which frequency in the Fourier series for $f(t)$ is closest to resonance for the system in Part (b).

Problem 5. (Heat equation with boundary and initial conditions)

For $0 \le x \le \pi$ and $t > 0$ we have

PDE: $u_t = u_{xx}$

BC: $u(0,t) = 0$ and $u(\pi, t) = 0$ for all $t > 0$.

IC: $u(x, 0) = f(x)$ for $0 \le x \le \pi$.

Showing all the steps clearly, use the separation of variables method to get the general solution $u(x,t)$ which satisfies both the PDE and BC.

Then give formulas for the Fourier coefficients of the solution which also satisfies the IC . These formulas will have to be given in terms of f.

Problem 6.
(a) Write down the wave equation with IC's and BC's for the string of length 1, with clamped ends, wave speed 2, initially at equilibrium, struck at time 0. Then derive the Fourier series solution using separation of variables.

(b) Give the explicit solution to the equation of Part (a) when the initial velocity is given by $f(x) = x$ on $0 < x < 1$ (as if that were possible!).

Problem 7.
Consider the following partial differential equation with boundary and initial conditions:

PDE: $u_t(x,t) + u(x,t) = u_{xx}(x,t)$; defined for $0 < x < 1$.

BC: $u(0,t) = 0, \quad u(1,t) = 0$.

IC: $u(x,0) = f(x)$.

(a) The separation of variables technique looks for solutions to the PDE of the form $u(x,t) = X(x)T(t)$. Give the ordinary DEs satisfied by X and T.

You do not have to solve these DEs.

(b) To make your life easier, we'll tell you that, in the usual notation, the only separated solutions satisfying the boundary conditions have $\lambda > 0$ and are of the form

$$X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \text{ and } T(t) = e^{-(1+\lambda)t}.$$

Of course, not all $\lambda > 0$ work. Find all the separated solutions to the PDE that satisfy the boundary conditions. Then give the general solution to the PDE with BC.

(c) Give the Fourier solution to PDE with BC and IC. Be sure to write down the integral formula for any coefficients used. (Since f is not specified you cannot compute the integrals.)

(d) We can add input to the PDE: $u_{xx} = u_t + u + xe^{-t}$.

A particular solution to this PDE also satisfying the BC of Part (a) is: $u_p(x,t) = \left(\frac{x^3}{6} - \frac{x}{6}\right)e^{-t}$.

What are all the solutions to this PDE which also satisfy the BC?

Problem 8.
Solve the wave equation with boundary and initial conditions.

PDE: $y_{tt} = y_{xx}$ for $0 \le x \le 1, \quad t > 0$

BC: $y(0,t) = 0,\ y(1,t) = 0$

IC: $y(x,0) = 0,\ y_t(x,0) = 30$.

Problem 9.
Solve the heat equation with insulated ends.

(Here's a problem that gives a cosine series so the $\lambda = 0$ case is important.)

PDE: $u_t = 3u_{xx}$ for $0 \le x \le 1,\ t > 0$

BC: $u_x(0,t) = 0,\ u_x(1,t) = 0$

IC: $u(x,0) = x$.

End of practice quiz