ES.1803 Practice Solutions – Quiz 7, Spring 2024

Problem 1.
Let $A = \begin{bmatrix} 2 & -3 \\ 2 & -2 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

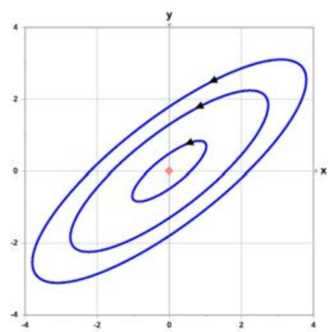
(iii) Sketch the trajectories

Solution: (i) Characteristic equation: $\lambda^2 + 6 = 0$. So the eigenvalues are $\lambda = \pm\sqrt{2}\,i$.

(ii) Pure imaginary eigenvalues mean the critical point is a center. This is an edge case stability-wise. Some call it a marginally dynamically stable critical point.

Because the lower left entry of A is positive, the loops turn counterclockwise. (This is because the tangent vector at (1,0) points up.)

(iii) Here's the phase portrait.



Problem 2.
Let $A = \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

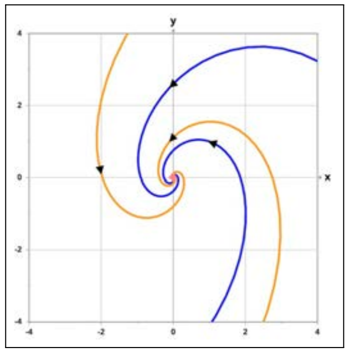
(iii) Sketch the trajectories

Solution: (i) Characteristic equation: $\lambda^2 + 3\lambda + 7 = 0$. So the eigenvalues are $\dfrac{-3 \pm \sqrt{19}\,i}{2}$.

(ii) Complex eigenvalues with negative real part mean the critical point is a spiral sink. This is a dynamically stable critical point.

Because the lower left entry of A is positive, the spirals turn counterclockwise. (This is because the tangent vector at (1,0) points up.)

(iii) Here's the phase portrait.



Problem 3.
Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

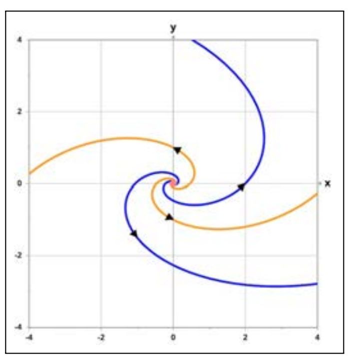
(iii) Sketch the trajectories

Solution: (i) Characteristic equation: $\lambda^2 - 2\lambda + 3 = 0$. So the eigenvalues are $1 \pm \sqrt{2}\, i$.

(ii) Complex eigenvalues with positive real part mean the critical point is a spiral source. This is a dynamically unstable critical point.

Because the lower left entry of A is positive, the spirals turn counterclockwise. (This is because the tangent vector at (1,0) points up.)

(iii) Here's the phase portrait.



Problem 4.
Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

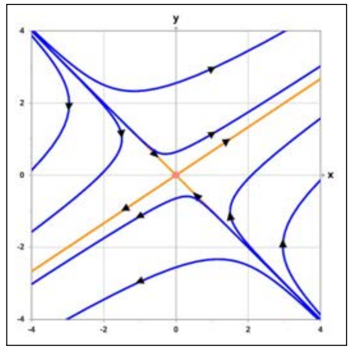
(iii) Sketch the trajectories

Solution: Solution: (i) Characteristic equation: $\lambda^2 - 3\lambda - 4 = 0$. So the eigenvalues are $4, -1$.

(ii) Real eigenvalues, one positive and one negative mean the critical point is a saddle. This is a dynamically unstable critical point.

For saddles, a qualitative phase portrait requires computing the eigenvectors. We find that an eigenvector corresponding to $\lambda = 4$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and one corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

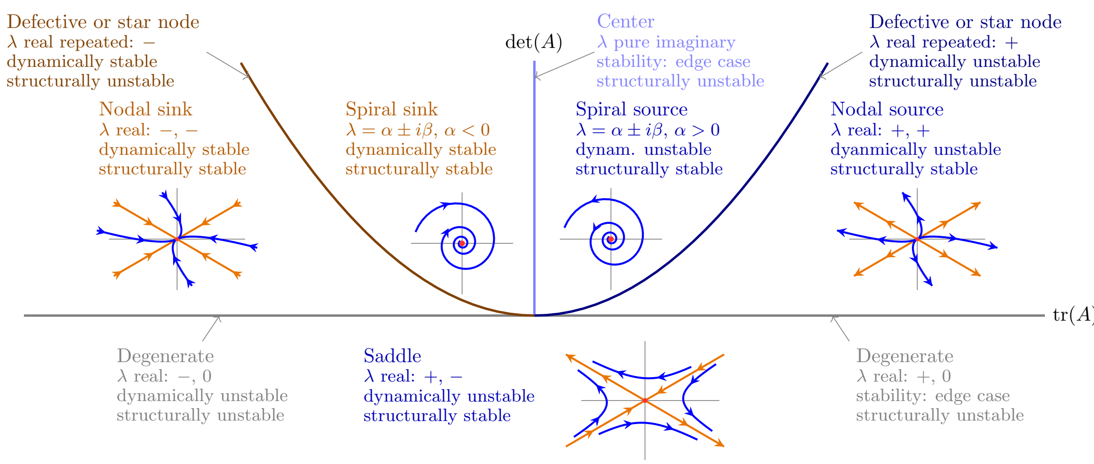
(iii) Here's the phase portrait.



Problem 5.

(a) *Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?*

Solution: Here is the diagram:



The open regions in the diagram all represent structurally stable systems. That is, nodal sources, spiral sources, nodal sinks, spiral sinks and saddles are all structurally stable. The lines represent structurally unstable systems, i.e., defective and star nodes, centers, degenerate systems.

(b) *Give the equation for the parabola in the diagram. Explain where is comes from.*

Solution: The characteristic equation is $\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0$. Therefore, the eigenvalues are

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}.$$

The parabola is the dividing line between real and imaginary root. That is it's where the discriminant (part under the square root) is 0. Its equation is

$$\mathrm{tr}(A)^2 - 4\det(A) = 0 \;\Leftrightarrow\; \det(A) = \frac{\mathrm{tr}(A)^2}{4}.$$

Problem 6.
Locate each of the following matrices on the trace-det diagram. Identify the type of critical point at the origin for the corresponding linear system $\mathbf{x}' = A\mathbf{x}$.

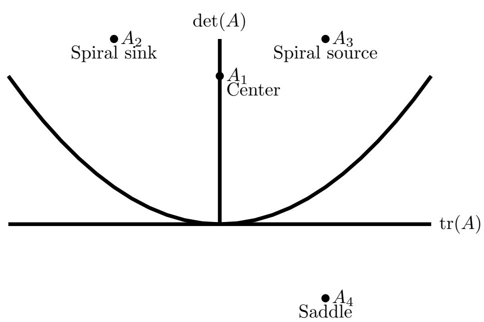
$$A_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

Solution: A_1: $\mathrm{tr}(A) = 0$, $\det(A) = 4$. This is on the positive vertical axis. The type of critical point is a center.

A_2: $\mathrm{tr}(A) = -2$, $\det(A) = 5$. This is in the second quadrant above the parabola $\det(A) = \mathrm{tr}(A)^2/4$. The type of critical point is a spiral sink.

A_3: $\mathrm{tr}(A) = 2$, $\det(A) = 5$. This is in the first quadrant above the parabola $\det(A) = \mathrm{tr}(A)^2/4$. The type of critical point is a spiral source.

A_4: $\mathrm{tr}(A) = 2$, $\det(A) = -2$. Since the determinant is negative, the type of critical point is a saddle.



Problem 7.
For each of the following linear systems, sketch phase portraits. Give the dynamic stability of the critical point at the origin. Give the structural stability of the system.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 1 \\ -4 & 10 \end{bmatrix}$

Solution: For all of these problems, we write the characteristic equation as

$$\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0, \qquad \text{where } A \text{ is the coefficient matrix.}$$

We only compute eigenvectors when there are real roots so we can draw the modes. For complex roots we simply determine the sense of the rotation.

We will not actually show the computations, just the results. All the pictures are at the end of the solution.

Eigenvalues: $\lambda = 6,\ 9$. Corresponding basic eigenvectors: $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Nodal source. The equilibrium at (0,0) is dynamically unstable. The system is structurally stable.

(b) $\mathbf{x}' = \begin{bmatrix} -7 & -3 \\ 3 & -17 \end{bmatrix}$

Eigenvalues: $\lambda = -8, -16$. Corresponding basic eigenvectors: $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Nodal sink. The equilibrium at (0,0) is dynamically asymptotically stable. The system is structurally stable.

(c) $\mathbf{x}' = \begin{bmatrix} 5 & 3 \\ 0 & -2 \end{bmatrix}$

Eigenvalues: $\lambda = 5, -2$. Corresponding basic eigenvectors: $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

Saddle. The equilibrium at (0,0) is dynamically unstable. The system is structurally stable.

(d) $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -5 & -1 \end{bmatrix}$

Eigenvalues: $\lambda = 2 \pm 4i$.

Spiral source (clockwise). The equilibrium at (0,0) is dynamically unstable. The system is structurally stable.

(e) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & -3 \end{bmatrix}$

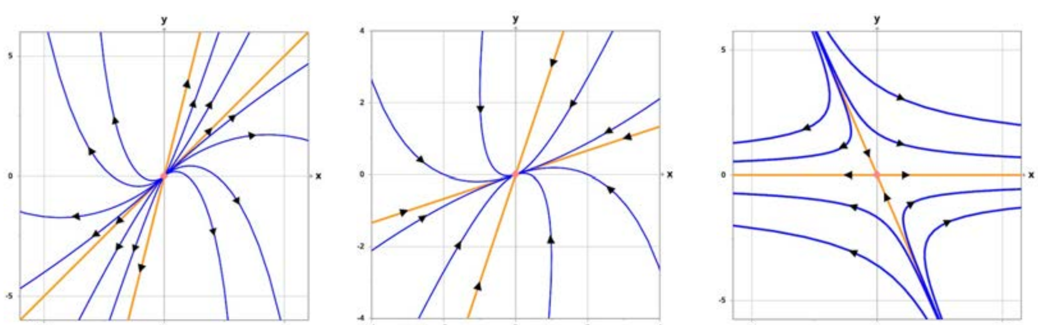
Eigenvalues: $\lambda = \pm i\sqrt{7}$.

Center (counterclockwise). The equilibrium at (0,0) is an edge case stablity-wise (or dynamically marginally stable). The system is structurally unstable.

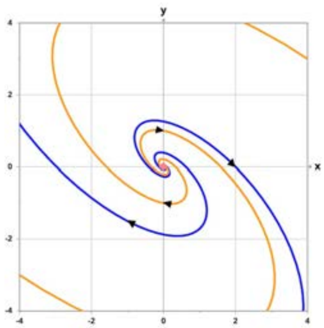
(f) $\mathbf{x}' = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{x}$

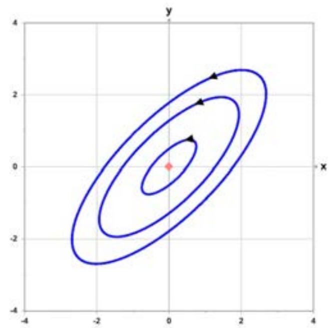
Eigenvalues: $\lambda = -2, -2$, Only one independent eigenvector $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

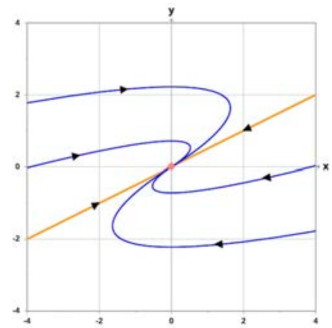
Defective nodal sink. The equilibrium at (0,0) is dynamically asymptotically stable. The system is structurally unstable.



(a) Nodal source (b) Nodal sink (c) Saddle


(d) Spiral source


(e) Center


(f) Defective nodal sink

Problem 8.
For the system of DEs $\mathbf{x}' = A_a\mathbf{x}$, *where* $A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$:

(a) *Find the range of the values of a for which the critical point at (0,0) will be:*
(i) a source node (ii) a sink node (iii) a saddle.

Solution: Characteristic equation: $P(\lambda) = \lambda^2 - 2a\lambda + a^2 - 1 = 0$.

Eigenvalues: $\lambda = a \pm 1$.

(i) $a > 1 \Rightarrow \lambda_1, \lambda_2 > 0 \Rightarrow$ source node.

(ii) $a < -1 \Rightarrow \lambda_1, \lambda_2 < 0 \Rightarrow$ sink node.

(iii) $-1 < a < 1 \Rightarrow \lambda_1 > 0\, \lambda_2 < 0 \Rightarrow$ saddle.

(b) *Choose a convenient value for a for each of the types above, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.*

Solution: Sketches appear below. We choose $a = 2,\ 0,\ -2$.

(i) $a = 2$
Eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = 1$.

Eigenvectors $\mathbf{v_1} = (1, 1)^T$ and $\mathbf{v_2} = (1, -1)^T$.

Normal modes: $\mathbf{x_1} = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(ii) $a = -2$
Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -3$.

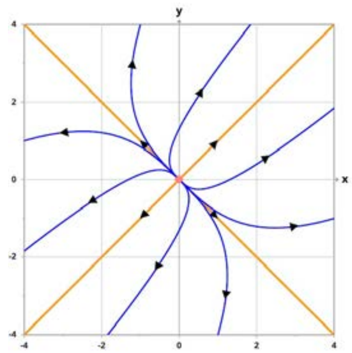
Eigenvectors $\mathbf{v_1} = (1, 1)^T$ and $\mathbf{v_2} = (1, -1)^T$.

Normal modes: $\mathbf{x_1} = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

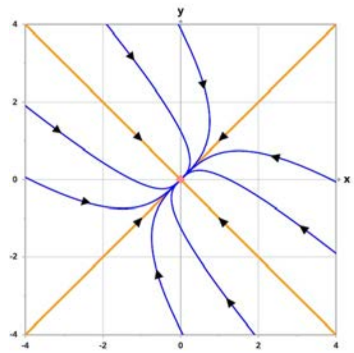
(iii) $a = 0$
Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -1$.

Eigenvectors $\mathbf{v_1} = (1, 1)^T$ and $\mathbf{v_2} = (1, -1)^T$.

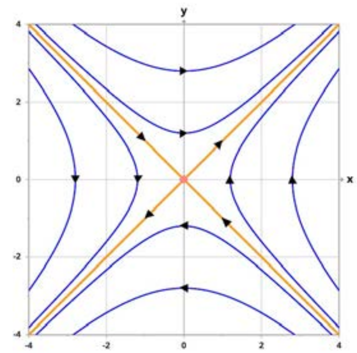
Normal modes: $\mathbf{x_1} = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



$a = 2$: nodal source



$a = -2$: nodal sink



$a = 0$: saddle

Problem 9.

For the DE system $x' = x - 2\,y \qquad y' = 4\,x - x^3$*:*

(a) *Compute the critical points of this system.*

Solution: Critical points: First we factor the expression for y':

$$y' = x(4 - x^2) = 0 \Rightarrow x = 0, \pm 2.$$

Trying these values in the equation $x' = x = 2y = 0$, we find the critical points are $(0, 0)$, $(2, 1)$, $(-2, -1)$.

(b) *For each critical point, find the type of critical point of the linear system which approximates this non-linear system. Give the linearized type and dynamic stability. Say what this tells you about the actual nonlinear system*

Solution: Jacobian: $J(x, y) = \begin{bmatrix} 1 & -2 \\ 4 - 3x^2 & 0 \end{bmatrix}$.

Critical point $(0, 0)$: $\quad J(0, 0) = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$

Characteristic equation: $P(\lambda) = \lambda^2 - \lambda + 8 = 0$.

Eigenvalues: $\frac{1}{2}(1 \pm \sqrt{-31})$.

So this is a linearized spiral source. The equilibrium is dynamically unstable.

Since spiral sources are structurally stable, the nonlinear system also looks like a spiral source near the critical point.

Direction: test at the point $(1, 0)$ in the plane the vector field $\mathbf{x}' = (1, 3)^T$, thus the spiral turns in a counterclockwise manner.

(Or we could simply note that the lower left-hand entry of the matrix is positive, so the trajectory is counterclockwise.)

Critical point $(2, 1)$: $\quad J(2, 1) = \begin{bmatrix} 1 & -2 \\ -8 & 0 \end{bmatrix}$

Characteristic equation: $P(\lambda) = \lambda^2 - \lambda - 16 = 0$.

Eigenvalues: $\frac{1}{2}(1 \pm \sqrt{65})$.

So this is a linearized saddle. The equilibrium is dynamically unstable

Since saddles are structurally stable, the nonlinear system also looks like a saddle near the critical point.

Call the Jacobian J. For Part (c) we'll want basic eigenvectors of $\text{Null}(J - \lambda I)$.

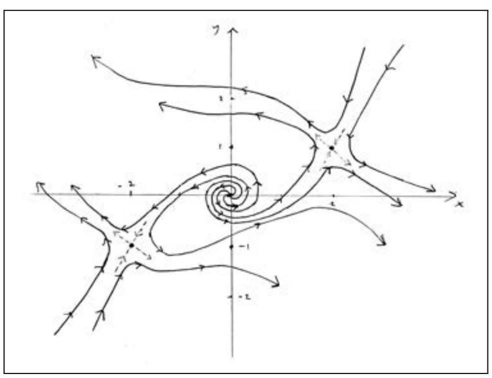
$\lambda = \dfrac{1+\sqrt{65}}{2}$: $\quad J - \lambda I = \begin{bmatrix} 1/2 - \sqrt{65}/2 & -2 \\ -8 & -1/2 - \sqrt{65}/2 \end{bmatrix}$. Basic eigenvaector: $\begin{bmatrix} 4 \\ 1 - \sqrt{65} \end{bmatrix} \approx \begin{bmatrix} 4 \\ -7.1 \end{bmatrix}$.

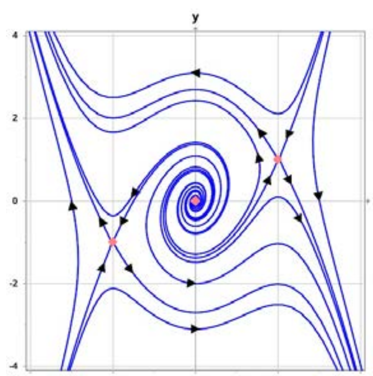
$\lambda = \dfrac{1-\sqrt{65}}{2} \quad J - \lambda I = \begin{bmatrix} 1/2 + \sqrt{65}/2 & -2 \\ -8 & -1/2 + \sqrt{65}/2 \end{bmatrix}$. Basic eigenvaector: $\begin{bmatrix} 4 \\ 1 + \sqrt{65} \end{bmatrix} \approx \begin{bmatrix} 4 \\ 9.1 \end{bmatrix}$.

Critical point $(-2,-1)$: $\quad J(-2,-1) = \begin{bmatrix} 1 & -2 \\ -8 & 0 \end{bmatrix}$, same Jacobian as $(2,1)$: saddle, dynamically unstable equilibrium, structurally stable.

(c) *Using the results of Part (b), compute the eigenvectors as needed. Then put it all together into a reasonable sketch of the phase plane portrait of this system. Is there more than one possiblity for the general shape and dynamic stability type of the trajectories around each of the critical points in this case? Why/why not?*

Solution: There is only one possibility for the general shape and dynamic stability type of the trajectories near each critical point since they are all structurally stable. Both a hand-drawn and computer drawn phase portrait are shown below.





Problem 10.
Same instructions as in previous problem for the DE system $x' = y \qquad y' = 2x - x^2$.

(a) Solution: Critical points: by inspection $(0,0)$ and $(2,0)$.

(b) Solution: Jacobian: $J(x,y) = \begin{bmatrix} 0 & 1 \\ 2 - 2x & 0 \end{bmatrix}$.

Critical point $(0,0)$: $\quad J(0,0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Characteristic equation: $P(\lambda) = \lambda^2 - 2 = 0$.

Eigenvalues: $\lambda = \pm\sqrt{2} \Rightarrow$ saddle.

Dynamically unstable equilibrium, structurally stable.

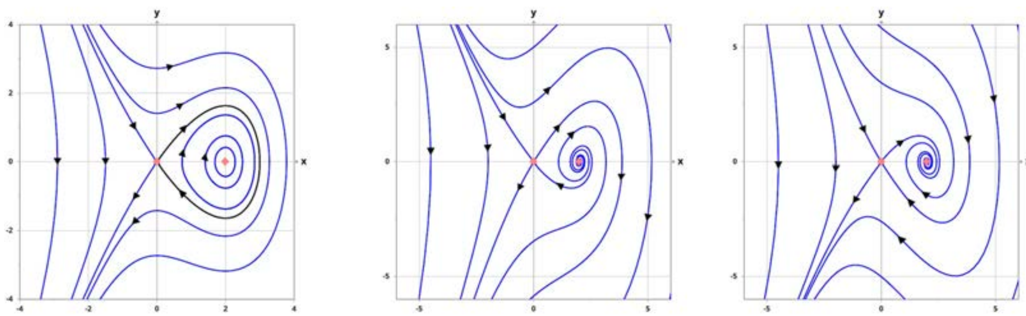
Critical point $(2,0)$: $\quad J(2,0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

Characteristic equation: $P(\lambda) = \lambda^2 + 2 = 0$.

Eigenvalues: $\pm i\sqrt{2} \Rightarrow$ center.

Dynamically an edge case, not structurally stable.

(c) Solution: There is only one possible type and general shape near the critical point at $(0,0)$ because it's structurally stable. But near the critical point at $(2,0)$ the non-linear system could look like a center, a spiral source or a spiral sink. By perturbing the system we got pplane to draw all 3 possibilities.



Actual system: center at (2,0) Spiral source at (2,0) Spiral sink at (2,0)

Problem 11.

(a) *Suppose that the DE system*

$$x' = x(x-2)^2 - xy \qquad y' = -y + 4xy$$

is used to describe the time rates of change of the population levels x, y for two interacting species. Will this interaction produce a sustainable long-term positive population level for both species? If so, what will this equilibrium value be? (If you get any positive values, assume that the units for x and y have been chosen so that these numbers are possible.)

Solution: Critical points: $y' = 0 \Rightarrow y(4x-1) = 0 \Rightarrow y = 0$ or $x = \frac{1}{4}$.

$y = 0 : x' = 0 \Rightarrow x(x-2)^2 = 0 \Rightarrow x = 0$ or $x = 2$.

$x = \frac{1}{4} : x' = 0 \Rightarrow y = \frac{49}{16}$.

Critical points are $(0,0)$, $(1/4, 49/16)$ and $(2,0)$.

Jacobian: $J(x,y) = \begin{bmatrix} 3x^2 - 8x + 4 - y & -x \\ 4y & -1 + 4x \end{bmatrix}$

Critical point $(1/4, 49/16)$: $\quad J(1/4, 49/16) = \begin{bmatrix} -7/8 & -1/4 \\ 49/4 & 0 \end{bmatrix}$.

Characteristic equation: $P(\lambda) = \lambda^2 + \frac{7}{8}\lambda + \frac{49}{16} = 0$.

Eigenvalues: $\frac{7}{16}(-1 \pm \sqrt{-15})$. (The exact values are not important to us, we just care the eigenvalues are complex with negative real part.) $\Rightarrow$ spiral sink.

Dynamically stable equilibrium, structurally stable.

Critical point $(0,0)$: $\quad J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

Eigenvalues: 4, -1 $\Rightarrow$ saddle.

Dynamically unstable equilibrium, structurally stable.

Critical point $(2,0)$: $\quad J(2,0) = \begin{bmatrix} 0 & -2 \\ 0 & 7 \end{bmatrix}$

Eigenvalues: 0, 7 $\Rightarrow$ line of critical points.

Not an isolated equilibrium, not structurally stable.

The answer to the question is yes, there is a long-term sustainable population level, i.e., positive stable equilibrium, for both species. It is given by the equilibrium values $x = 1/4$ and $y = 49/16$.

(b) *Observe that if the second species was not present (i.e., $y = 0$) then the first species (x) would be modeled by the autonomous DE*

$$x' = x(x-2)^2$$

Comparing the stabilty analysis in this case to the outcome found for the two-species model in Part (a): what can one see – directly from the DE rate statements – about the way in which the interaction with the second species (y) changed the growth and stabilty/instablilty behavior of the population size x when it was the only species present?

(Note: just a few lines of explanation here should be enough to get the main point across.)

Solution: The critical points for the equation $x' = x(x-2)^2$ are 0 and 2. The positive equilibrium (at $x = 2$) is semistable: if the population goes above 2 it explodes. In the two species model the population x is kept in check since if it grows the growth rate of the predator-type species y increases and the growth rate of x decreases. (This shows the result is plausible.) To be sure that y increases sufficiently rapidly to keep the population x in check, you need to do the analysis as in Part (a). The key is the interaction coefficient (the 4 in the $4xy$ term). If it is made sufficiently small then the population x can explode as in the single species model.

End of practice quiz solutions