ES.1803 Practice Questions – Quiz 7, Spring 2024
Covers Topics 27-30

This will probably take longer than 1 hour; the actual quiz will be considerably shorter. Not every possible question is covered. For more questions, you can also look at the in-class problems, problem section worksheets and psets.

Be wise in carrying out computations. If you get a problem to a point where you are sure –*emphasize sure*– you can carry out the computation, then stop and go on to the next problem. At the end you can check the solutions to see that you had correctly set up the computation.

Problem 1.
Let $A = \begin{bmatrix} 2 & -3 \\ 2 & -2 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

(iii) Sketch the trajectories

Problem 2.
Let $A = \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

(iii) Sketch the trajectories

Problem 3.
Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

(iii) Sketch the trajectories

Problem 4.
Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$.

(i) Find the eigenvalues.

(ii) Give the name and dynamic stability for the critical point at the origin.

(iii) Sketch the trajectories

Problem 5.
(a) Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?

(b) Give the equation for the parabola in the diagram. Explain where is comes from.

Problem 6.
Locate each of the following matrices on the trace-det diagram. Identify the type of critical point at the origin for the corresponding linear system $\mathbf{x}' = A\mathbf{x}$.

$$A_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

Problem 7.
For each of the following linear systems, sketch phase portraits. Give the dynamic stability of the critical point at the origin. Give the structural stability of the system.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 1 \\ -4 & 10 \end{bmatrix}$

(b) $\mathbf{x}' = \begin{bmatrix} -7 & -3 \\ 3 & -17 \end{bmatrix}$

(c) $\mathbf{x}' = \begin{bmatrix} 5 & 3 \\ 0 & -2 \end{bmatrix}$

(d) $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -5 & -1 \end{bmatrix}$

(e) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & -3 \end{bmatrix}$

(f) $\mathbf{x}' = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{x}$

Problem 8.
For the system of DEs $\mathbf{x}' = A_a\mathbf{x}$, where $A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$:

(a) Find the range of the values of a for which the critical point at (0,0) will be:

(i) a source node (ii) a sink node (iii) a saddle.

(b) Choose a convenient value for a for each of the types above, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.

Problem 9.
For the DE system $x' = x - 2\,y \qquad y' = 4\,x - x^3$:

(a) Compute the critical points of this system.

(b) For each critical point, find the type of critical point of the linear system which approximates this non-linear system. Give the linearized type and dynamic stability. Say what this tells you about the actual nonlinear system

(c) Using the results of Part (b), compute the eigenvectors as needed. Then put it all together into a reasonable sketch of the phase plane portrait of this system. Is there more than one possiblity for the general shape and dynamic stability type of the trajectories around each of the critical points in this case? Why/why not?

Problem 10.
Same instructions as in previous problem for the DE system $x' = y \qquad y' = 2\,x - x^2$.

Problem 11.
(a) Suppose that the DE system

$$x' = x(x-2)^2 - xy \qquad\qquad y' = -y + 4xy$$

is used to describe the time rates of change of the population levels x, y for two interacting species. Will this interaction produce a sustainable long-term positive population level for *both* species? If so, what will this equilibrium value be? (If you get any positive values, assume that the units for x and y have been chosen so that these numbers are possible.)

(b) Observe that if the second species was not present (i.e., $y = 0$) then the first species (x) would be modeled by the autonomous DE

$$x' = x(x-2)^2$$

Comparing the stabilty analysis in this case to the outcome found for the two-species model in Part (a): what can one see – directly from the DE rate statements – about the way in which the interaction with the second species (y) changed the growth and stabilty/instablilty behavior of the population size x when it was the only species present?

(Note: just a few lines of explanation here should be enough to get the main point across.)

End of practice quiz

MIT OpenCourseWare
https://ocw.mit.edu

ES.1803 Differential Equations
Spring 2024

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.