ES.1803 Problem Section 10, Spring 2024 Solutions

Integral table

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}$$
$$\int t\sin(\omega t)\,dt = -\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}$$
$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}$$
$$\int t^2\sin(\omega t)\,dt = -\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}$$
$$\int e^t\cos(\omega t)\,dt = \frac{e^t\cos(\omega t)}{1+\omega^2} + \frac{\omega e^t\sin(\omega t)}{1+\omega^2}$$
$$\int e^t\sin(\omega t)\,dt = -\frac{\omega e^t\cos(\omega t)}{1+\omega^2} + \frac{e^t\sin(\omega t)}{1+\omega^2}$$
$$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$
$$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$
$$\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$$
$$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$
$$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$
$$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$

Problem 20.1. *Compute the following integrals.*

(a) $\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\,dt$

Solution: Both spikes are inside the interval of integration. So the integral equals 4.

(b) $\int_{1}^{5} \delta(t) + 3\delta(t-2) + 4\delta(t-6)\,dt.$

Solution: Only the spike at $t = 2$ is inside the interval of integration. So the integral equals 3.

Problem 20.2. *Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC*

Solution: Rest IC means that for $t < 0$ we have $x(t) = 0$.

We work on the intervals between the impulses one at a time.

For $t < 0$: The DE is $x' + 2x = 0$, with $x(0^-) = 0$. The solution is $x(t) = 0$.

For $0 < t < 3$: We are given the pre-initial condition $x(0^-) = 0$.

The impulse at $t = 0$ causes x to jump by one unit. During the rest of the interval the input is 0. So, for $0 < t < 3$, we have

$$x' + 2x = 0, \quad x(0^+) = x(0^-) + 1 = 1.$$

Solving, we get $x(t) = x(0^+)e^{-2t} = e^{-2t}$.

The end condition for this interval is $x(3^-) = e^{-6}$.

For $3 < t$: From the previous interval we have the pre-initial condition $x(3^-) = e^{-6}$.

The impulse at $t = 3$ causes a unit jump in x, so $x(3^+) = x(3^-) + 1 = e^{-6} + 1$. Again, during the rest of the interval the input is 0. So we have

$$x' + 2x = 0, \quad x(3^+) = e^{-6} + 1.$$

Solving, we get $x(t) = x(3^+)e^{-2(t-3)} = (e^{-6} + 1)e^{-2(t-3)}$.

Putting the cases together, we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 < t < 3 \\ (e^{-6} + 1)e^{-2(t-3)} & \text{for } t > 3. \end{cases}$$

Problem 20.3. *(Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.*

Solution: The key is understanding what jump the input $5\delta(t)$ causes at $t = 0$. In this case, since the leading coefficient is 4, $5\delta(t)$ causes a jump of 5/4 unit in x'.

Rest IC means $x(0^-) = 0$, $x'(0^-) = 0$.

For $t < 0$: The DE with initial conditions is

$$4x'' + x = 0; \qquad x(0^-) = 0,\ x'(0^-) = 0.$$

The solution on this interval is $x(t) = 0$.

For $t > 0$: The pre-initial conditions are $x(0^-) = 0, x'(0^-) = 0$. The input $5\delta(t)$ is an impulse which produces post-initial conditions

$$x(0^+) = 0, \quad x'(0^+) = x'(0^-) + 5/4 = 5/4.$$

After the impulse the input is 0. So the DE with initial conditions is

$$4x'' + x = 0; \qquad x(0^+) = 0,\ x'(0^+) = 5/4.$$

The characteristic roots are $\pm 2i$. So the general solution to the DE is

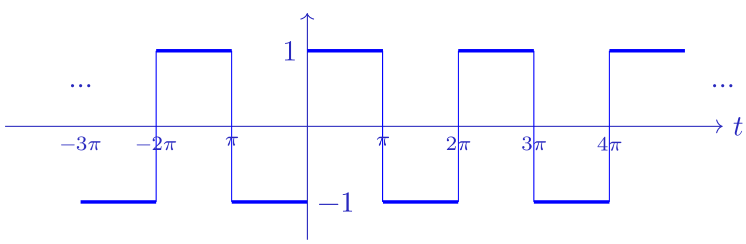
$$x(t) = c_1 \cos(t/2) + c_2 \sin(t/2).$$

We need to find c_1 and c_2 to match the post-initial conditions. The algebra yields $c_1 = 0$, $c_2 = 5/2$.

Putting the cases together, we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{5}{2}\sin(t/2) & \text{for } t > 0. \end{cases}$$

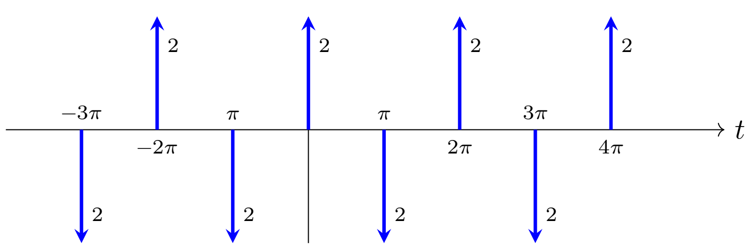
Problem 20.4. Derivative of a square wave
The graph below is of a function $\text{sq}(t)$ *(called a square wave). Compute and graph its generalized derivative.*



Graph of $\text{sq}(t)$ = *square wave*

Solution: The function alternates every π seconds between ± 1. The derivative $\text{sq}'(t)$ is clearly 0 everywhere except at the jumps. A jump of $+2$ gives a (generalized) derivative of 2δ and a jump of -2 gives a (generalized) derivative of -2δ. Thus we have

$$\text{sq}'(t) = ... + 2\delta(t + 2\pi) - 2\delta(t + \pi) + 2\delta(t) - 2\delta(t - \pi) + 2\delta(t - 2\pi) - 2\delta(t - 3\pi) + ...$$



Graph of $\text{sq}'(t)$ = impulse train

Note that we put the weight of each delta function next to it. Conventions vary, here we used the convention that $-2\delta(t)$ is represented by a downward arrow with the weight 2 next to it. That is, the sign is represented by the direction of the arrow, so the weight is positive.

Problem 21.5. *Compute the Fourier series for the period 2π triangle wave*

$$f(t) = |t| \text{ for } -\pi < t < \pi.$$

Solution: We have $L = \pi$. The integrals below were found using the integral table.

Since $|t|\cos(nt)$ is an even function, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \begin{cases} -\frac{4}{\pi n^2} & \text{for } n \text{ odd} \\ \pi & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$$

Likewise, $|t|\sin(nt)$ is an odd function, so $b_n = \frac{1}{\pi}\int_{\pi}^{\pi} |t|\sin(nt)\,dt = 0$. Thus,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}.$$

Problem 21.6. *For each of the following:*

(i) Find the Fourier series (no integrals needed)

(ii) Identify the fundamental frequency and corresponding base frequency.

(iii) Identify the Fourier coefficients a_n and b_n

(a) $\cos(2t)$

Solution: (i) Fourier series: $\cos(2t)$.

(ii) Fundamental frequency = 2. Base period $= \frac{2\pi}{2} = \pi$.

(iii) $a_1 = 1$ all other coefficients are 0.

(b) $3\cos(2t - \pi/6)$

Solution: (i) Fourier series: $3\cos(\pi/6)cos(2t) + 3\sin(\pi/6)\sin(2t) = \frac{3\sqrt{3}}{2}\cos(2t) + \frac{3}{2}\sin(2t)$.

(ii) Fundamental frequency = 2. Base period $= \pi$.

(iii) $a_1 = 3\sqrt{3}/2$, $b_1 = 3/2$, all others are 0.

(c) $\cos(t) + 2\cos(5t)$

Solution: (i) Fourier series: $\cos(t) + 2\cos(5t)$.

(ii) Fundamental frequency = 1. Base period $= 2\pi$.

(iii) $a_1 = 1$, $a_5 = 2$, all others are 0.

(d) $\cos(3t) + \cos(4t)$

Solution: (i) Fourier series: $\cos(3t) + \cos(4t)$.

(ii) Fundamental frequency = 1. (Note: Every frequency must be a multiple of the fundamental frequency.) Base period = 2π. (Note: 2π is the smallest common period for all the terms.)

(iii) $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, all others are 0.

Problem 22.7. *The function $f(t)$ has period π. Over the interval $0 \le x < \pi$ we have $f(t) = \sin(t)$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$*

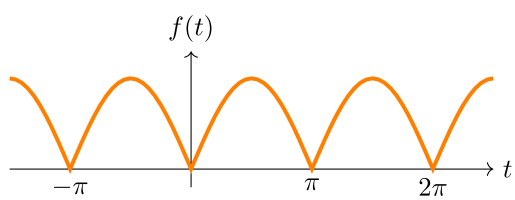
Solution: This is an even function, so we only need to compute the cosine coefficients (a_n). We don't show all the details of the integrations. An integral table will help here

We have the half-period $L = \pi/2$. In this case, I think it is easiest to integrate over a full period $[0, \pi]$ rather than use the doubling trick for even functions.

$$a_0 = \frac{1}{\pi/2}\int_0^\pi \sin(t)\,dt = -\frac{2}{\pi}\left[\cos(t)\right]_0^\pi = \frac{4}{\pi}$$

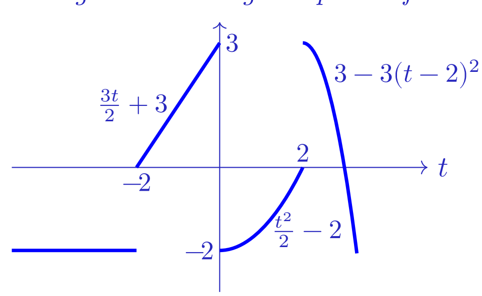
$$a_n = \frac{1}{\pi/2}\int_0^\pi \sin(t)\cos(2nt)\,dt = \frac{2}{\pi}\cdot\frac{1}{2}\left(-\frac{\cos((2n+1)t)}{2n+1} + \frac{\cos((2n-1)t)}{2n-1}\right]_0^\pi = -\frac{4}{\pi(4n^2-1)}$$

So, $f(t) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2nt)}{4n^2-1}$



Extra problems if time.

Problem 20.8. *The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.*



Solution: The formula for the function is

$$f(t) = \begin{cases} -2 & \text{for } t < -2 \\ \frac{3t}{2} + 3 & \text{for } -2 < t < 0 \\ \frac{t^2}{2} - 2 & \text{for } 0 < t < 2 \\ 3 - 3(t-2)^2 & \text{for } 2 < t. \end{cases}$$

We have to take the regular derivative and add delta functions at the jump discontinuities.

$$f'(t) = \underbrace{2\delta(t+2) - 5\delta(t) + 3\delta(t-3)}_{\text{singular part}} + \underbrace{\begin{cases} 0 & \text{for } t < -2 \\ \frac{3}{2} & \text{for } -2 < t < 0 \\ t & \text{for } 0 < t < 2 \\ -6(t-2) & \text{for } 2 < t. \end{cases}}_{\text{regular part}}$$

Problem 20.9. *Compute the following integrals.*

(a) $\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\,dt.$

Solution: All the spikes are inside the interval of integral so the integral is

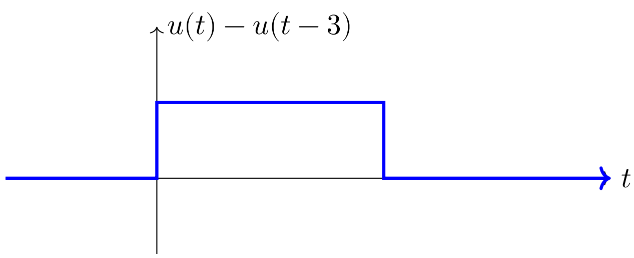
$$\cos(0) + \sin(\pi) + \cos(2\pi) = 2.$$

(b) $\int \delta(t)\,dt$. *(Indefinite integral)*

Solution: $u(t) + C$. (By definition $u' = \delta$.)

(c) $\int \delta(t) - \delta(t-3)\,dt$. *Graph the solution*

Solution: $u(t) - u(t-3) + C$.



Problem 20.10. *Solve $x' + 2x = \delta(t)$ with rest IC*

Solution: Rest IC means that for $t < 0$ we have $x(t) = 0$.

We are given the pre-initial condition $x(0^-) = 0$.

For $t < 0$: On this interval the input $\delta(t) = 0$. So the DE with initial conditions is

$$x' + 2x = 0; \qquad x(0^-) = 0.$$

The solution to the DE is $x(t) = ce^{-2t}$. The initial condition implies $x(0^-) = c = 0$. So, $\boxed{\text{for } t < 0,\ x(t) = 0.}$

For $t > 0$: The impulse at $t = 0$ causes x to jump by one unit. After that the input is 0. So, for $t > 0$, we have

$$x' + 2x = 0; \quad x(0^+) = 1.$$

Solving we get $x(t) = ce^{-2t}$. The initial condition implies $x(0^+) = c = 1$. So, $\boxed{\text{for } t > 0,\ x(t) = e^{-2t}}$.

Putting the cases together, we have:

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 < t \end{cases}$$

Problem 20.11. **(a)** *Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.*

Solution: Rest IC means $x(0^-) = 0$, $x'(0^-) = 0$.

On $t < 0$: The differential equation with initial conditions is

$$2x'' + 8x' + 6x = 0; \qquad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

The solution to this is $x(t) = 0$.

On $t > 0$: The impulse at $t = 0$ causes a jump in x'. That is, we have post-initial conditions $x(0^+) = 0$, $x'(0^+) = x'(0^-) + 1/2 = 1/2$.

So the differential equation with initial conditions is

$$2x'' + 8x' + 6x = 0; \qquad x(0^+) = 0,\ \dot{x}(0^+) = 1/2.$$

The characteristic roots are -1, -3. So the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

We find c_1 and c_2 to match the post-initial conditions: $c_1 = 0$, $c_2 = 1/2$. The complete solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{-t}}{4} - \frac{e^{-3t}}{4} & \text{for } 0 < t \end{cases}$$

(b) *Plug your solution into the DE and verify that it is correct*

Solution: We have to take two derivatives of x. Since $x(t)$ has no jump at $t = 0$, the generalized derivative has only a regular part.

$$x'(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{-e^{-t}}{4} + \frac{3e^{-3t}}{4} & \text{for } t > 0. \end{cases}$$

Since $x'(0^-) = 0$ and $x'(0^+) = 1/2$, x'' has a singular part:

$$x''(t) = \frac{\delta(t)}{2} + \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{-t}}{4} - \frac{9e^{-3t}}{4} & \text{for } t > 0. \end{cases}$$

Thus,

$$2x'' + 8x' + 6x = \delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ 2\left(\frac{e^{-t}}{4} - \frac{9e^{-3t}}{4}\right) + 8\left(\frac{-e^{-t}}{4} + \frac{3e^{-3t}}{4}\right) + 6\left(\frac{e^{-t}}{4} - \frac{e^{-3t}}{4}\right) & \text{for } t > 0 \end{cases}$$
$$= \delta(t).$$

So the solution checks out. Notice how, in the algebra, the jump of 1/2 in x' resulted in the $\delta(t)$ term when we plugged x into the DE.

Problem 20.12. *Solve $x' + 3x = \delta(t) + e^{2t}u(t) + 2\delta(t-4)$ with rest IC.*

(The $u(t)$ is there to make sure the input is 0 for $t < 0$.)

Solution: When part of the input is a regular function, you have to organize the work carefully. Here are two ways to do it.

Method 1. Solve in cases. Each delta function adds another case.

(Case 1) $t < 0$: $x' + 3x = 0$ with rest IC $\Rightarrow x(t) = 0$.

(Case 2) $0 < t < 4$. Pre-initial conditions $x(0^-) = 0$. The delta function gives post-initial conditions: $x(0^+) = 1$.

On this interval the DE is $x' + 3x = e^{2t}$.

Using the ERF, we get the general solution $x(t) = \frac{e^{2t}}{5} + C_2e^{-3t}$

The post-initial condition gives $x(0^+) = 1 = 1/5 + C_2$, so $\boxed{C_2 = 4/5.}$

(Case 3) $t > 4$. Using Case 2, the pre-initial conditions are $x(4^-) = \dfrac{e^8}{5} + \dfrac{4e^{-12}}{5}$. So the input $2\delta(t-4)$ gives post-initial conditions $x(4^+) = x(4^-) + 2 = \dfrac{e^8}{5} + \dfrac{4e^{-12}}{5} + 2$.

On this interval, the DE is the same as in Case 2: $x' + 3x = e^{2t}$. So we have the same general solution:

$$x(t) = \frac{e^{2t}}{5} + C_3e^{-3t}.$$

Using the post-initial conditions, we solve for C_3:

$$\frac{e^8}{5} + C_3e^{-12} = \frac{e^8}{5} + \frac{4e^{-12}}{5} + 2 \quad \Rightarrow \boxed{C_3 = 2e^{12} + \frac{4}{5}}.$$

Putting the cases together:

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{2t}}{5} + C_2e^{-3t} & \text{for } 0 < t < 4 \\ \frac{e^{2t}}{5} + C_3e^{-3t} & \text{for } t > 4 \end{cases}$$

where C_2 and C_3 are boxed above.

Method 2. We find particular solutions for each of the input pieces with rest initial conditions. Then we use superposition to find the solution we want.

We do the solving for each piece very quickly. You can fill in the details.

(i) Solve $x_1' + 3x_1 = \delta(t)$ with rest IC.

For $t > 0$: The post-initial conditions are $x_1(0^+) = 1$. The DE is $x_1' + 3x_1$. We have the solution:

$$x_1(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-3t} & \text{for } t > 0. \end{cases}$$

(ii) Solve $x_2' + 3x_2 = u(t)e^{2t}$ with rest IC.

We use the exponential response formula plus the general homogeneous solution.

For $t > 0$, $x_2(t) = \dfrac{e^{2t}}{5} + Ce^{-3t}$. The initial condition is $x_2(0) = 0$. This allows us to find $C = -1/5$. So,

$$x_2(t) = \begin{cases} 0 & \text{for } t < 0 \\ , \frac{e^{2t}}{5} - \frac{e^{-3t}}{5} & \text{for } t > 0. \end{cases}$$

(iii) Solve $x_3' + 3x_3 = 2\delta(t-4)$ with rest IC.

We have pre and post-initial conditions $x_3(4^-) = 0$, $x_3(4^+) = 2$. Solving we get

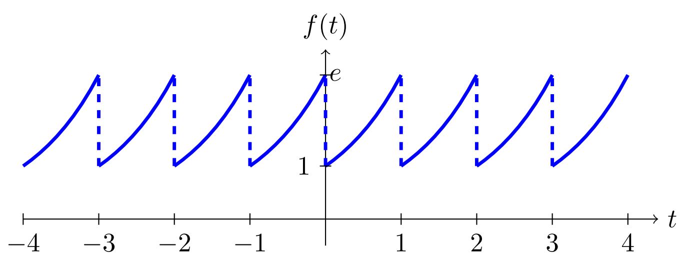
$$x_3(t) = \begin{cases} 0 & \text{for } t < 4 \\ 2e^{-3(t-4)} & \text{for } t > 4. \end{cases}$$

Note: You should verify for yourself, that, since each piece satisfies the rest initial condition, so does their sum. That is, we have homogeneous IC. If the IC was inhomogeneous, we would want just one of the pieces to satisfy the inhomogeneous IC and the others to satisfy the homogeneous IC.

Problem 21.13. *Consider the period 1 function given by $f(t) = e^t$ on $(0, 1)$.*

(a) *Graph the function.*

Solution: Note that the interval is open at both ends. We don't define $f(t)$ at the endpoints.



(b) *What would you expect about the decay rate of the Fourier coefficients?*

Solution: Since the function has jumps, we expect that one of a_n or b_n decays like $1/n$.

(c) *Compute the Fourier series. The integral table provided might help.*

Solution: First note that $L = 1/2$. None of our tricks work here, so we just use straightforward integration. We do our integrals over one full period: 0 to 1.

The given integral table is helpful.

$a_0 = 2\displaystyle\int_0^1 e^t\,dt = 2(e-1).$

$$a_n = 2\int_0^1 e^t\cos(2n\pi t)\,dt = 2\left[\frac{e^t\cos(2n\pi t)}{1+(2n\pi)^2} + \frac{2n\pi e^t\sin(2n\pi t)}{1+(2n\pi)^2}\right]_0^1 = \frac{2(e-1)}{1+(2n\pi)^2}$$

$$b_n = 2\int_0^1 e^t\sin(2n\pi t)\,dt = 2\left[\frac{e^t\sin(2n\pi t)}{1+(2n\pi)^2} - \frac{2n\pi e^t\cos(2n\pi t)}{1+(2n\pi)^2}\right]_0^1 = \frac{-4n\pi(e-1)}{1+(2n\pi)^2}$$

(a_n decays like $1/n^2$, b_n decays like $1/n$.) The Fourier series for f is

$$f(t) = (e-1) + \sum_{n=1}^{\infty}\frac{2(e-1)}{1+(2n\pi)^2}\cos(2n\pi t) + \sum_{n=1}^{\infty}\frac{-4n\pi(e-1)}{1+(2n\pi)^2}\sin(2n\pi t).$$

Problem 21.14. *Compute the Fourier series for the odd, period 2π, amplitude 1 square wave.*

Solution: The function is $f(t) = \begin{cases} -1 & \text{for } -\pi < t < 0 \\ 1 & \text{for } 0 < t < \pi \end{cases}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nt)\,dt + \int_0^{\pi}\cos(nt)\,dt\right) = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0 \text{ (by considering the area under the graph.)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nt)\,dt + \int_0^{\pi}\sin(nt)\,dt\right) = \frac{1-\cos(n\pi)}{n\pi} - \frac{\cos(n\pi)-1}{n\pi}.$$

Using $\cos(n\pi) = (-1)^n$, we get $b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ 4/n\pi & \text{if } n \text{ odd} \end{cases}$ So, $\boxed{f(t) = \sum_{n\text{ odd}} \frac{4\sin(nt)}{n\pi}.}$

Problem 22.15. **(a)** *Compute the Fourier series for the even, period 2π function, with $f(t) = \pi t - t^2$ on $[0,\pi]$. The integral table provided should help.*

Solution: Since $f(t)$ is even, $b_n = 0$.

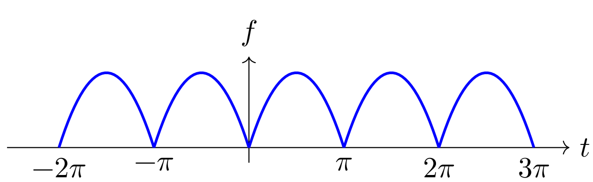
Using the integral table to compute the integrals, we find

$$a_n = \begin{cases} \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)\cos(nt)\,dt = -4/n^2 & \text{for } n \text{ even, } n \neq 0 \\ \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)\cos(nt)\,dt = 0 & \text{for } n \text{ odd} \\ \frac{2}{\pi}\int_0^{\pi}\pi t - t^2\,dt = \pi^2/3 & \text{for } n = 0. \end{cases}$$

So, $f(t) = \dfrac{\pi^2}{6} - 4\displaystyle\sum_{n\text{ even}}\frac{\cos(nt)}{n^2}$.

(b) *Carefully sketch the graph of the Fourier series.*

The function $f(t)$ is continuous at all t, so the Fourier series converges to $f(t)$

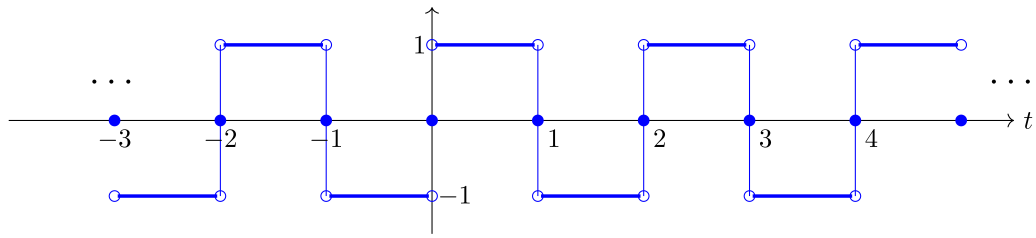


(c) *Challenge: Can you explain why the odd cosine coefficients are 0?*

Solution: This is really a period π function so its Fourier series has fundamental angular frequency 2.

Problem 22.16. *Let $f(t)$ be the odd, period 2, amplitude 1 square wave. Carefully sketch the graph of the Fourier series.*

Solution: The key to the sketch is to put dots at the midpoint of each jump and open circles at the ends of each line segment.



Problem 22.17. *Recall the Fourier series for the period 2π triangle wave tri(t), where tri$(t) = |t|$ for $-\pi \le t \le \pi$:*

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}.$$

Set $t = 0$ and show $\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$. (This is only for fun, we will not test on this sort of problem.)

Solution: We know $\text{tri}(0) = 0$. Putting $t = 0$ in the Fourier series gives

$$\text{tri}(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} = 0.$$

A small amount of algebra now shows that $\displaystyle\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}..$