ES.1803 Problem Section 10, Spring 2024

Integral table

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}$$

$$\int t\sin(\omega t)\,dt = -\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}$$

$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}$$

$$\int t^2\sin(\omega t)\,dt = -\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}$$

$$\int e^t\cos(\omega t)\,dt = \frac{e^t\cos(\omega t)}{1+\omega^2} + \frac{\omega e^t\sin(\omega t)}{1+\omega^2}$$

$$\int e^t\sin(\omega t)\,dt = -\frac{\omega e^t\cos(\omega t)}{1+\omega^2} + \frac{e^t\sin(\omega t)}{1+\omega^2}$$

$$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

$$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

$$\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$$

$$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$

$$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$

$$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$

Problem 20.1. Compute the following integrals.

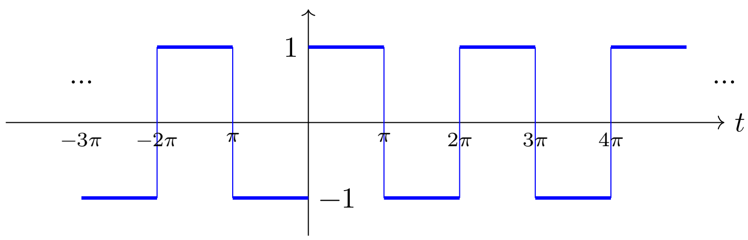
(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\,dt$

(b) $\displaystyle\int_{1}^{5} \delta(t) + 3\delta(t-2) + 4\delta(t-6)\,dt.$

Problem 20.2. Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC

Problem 20.3. (Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.

Problem 20.4. **Derivative of a square wave**
The graph below is of a function $\text{sq}(t)$ (called a square wave). Compute and graph its generalized derivative.



Graph of $\text{sq}(t)$ = square wave

Problem 21.5. Compute the Fourier series for the period 2π triangle wave

$$f(t) = |t| \text{ for } -\pi < t < \pi.$$

Problem 21.6. For each of the following:

(i) Find the Fourier series (no integrals needed)

(ii) Identify the fundamental frequency and corresponding base frequency.

(iii) Identify the Fourier coefficients a_n and b_n

(a) $\cos(2t)$

(b) $3\cos(2t - \pi/6)$

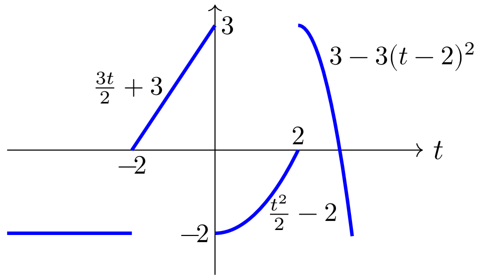
(c) $\cos(t) + 2\cos(5t)$

(d) $\cos(3t) + \cos(4t)$

Problem 22.7. The function $f(t)$ has period π. Over the interval $0 \le x < \pi$ we have $f(t) = \sin(t)$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$

Extra problems if time.

Problem 20.8. The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.



Problem 20.9. Compute the following integrals.

(a) $\displaystyle\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\,dt.$

(b) $\displaystyle\int \delta(t)\,dt.$ (Indefinite integral)

(c) $\displaystyle\int \delta(t) - \delta(t-3)\,dt.$ Graph the solution

Problem 20.10. Solve $x' + 2x = \delta(t)$ with rest IC

Problem 20.11. **(a)** Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.

(b) Plug your solution into the DE and verify that it is correct

Problem 20.12. Solve $x' + 3x = \delta(t) + e^{2t}u(t) + 2\delta(t-4)$ with rest IC.

(The $u(t)$ is there to make sure the input is 0 for $t < 0$.)

Problem 21.13. Consider the period 1 function given by $f(t) = e^t$ on $(0,1)$.

(a) Graph the function.

(b) What would you expect about the decay rate of the Fourier coefficients?

(c) Compute the Fourier series. The integral table provided might help.

Problem 21.14. Compute the Fourier series for the odd, period 2π, amplitude 1 square wave.

Problem 22.15. **(a)** Compute the Fourier series for the even, period 2π function, with $f(t) = \pi t - t^2$ on $[0,\pi]$. The integral table provided should help.

(b) Carefully sketch the graph of the Fourier series.

(c) Challenge: Can you explain why the odd cosine coefficients are 0?

Problem 22.16. Let $f(t)$ be the odd, period 2, amplitude 1 square wave. Carefully sketch the graph of the Fourier series.

Problem 22.17. Recall the Fourier series for the period 2π triangle wave $\text{tri}(t)$, where $\text{tri}(t) = |t|$ for $-\pi \leq t \leq \pi$:

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\,\text{odd}} \frac{\cos(nt)}{n^2}.$$

Set $t = 0$ and show $\sum_{n\,\text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$. (This is only for fun, we will not test on this sort of problem.)