# ES.1803 Problem Section 11, Spring 2024

**Integral table**

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}$$

$$\int t\sin(\omega t)\,dt = -\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}$$

$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}$$

$$\int t^2\sin(\omega t)\,dt = -\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}$$

$$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

$$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

$$\int \cos(at)\sin(bt)\,dt = -\frac{1}{2}\left[\frac{\cos((a+b)t)}{a+b} - \frac{\cos((a-b)t)}{a-b}\right]$$

$$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$

$$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$

$$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$

**Problem 23.1.** Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.

**Problem 23.2.** Find Fourier cosine series for $\sin(x)$ on $[0, \pi]$.

**Problem 24.3. (a)** Solve $x'' + 2x' + 9x = g(t)$, where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$. Find the Fourier series of $g$ by using $g(t) = f(\pi t)/\pi$, where $f$ is the standard period $2\pi$ triangle wave $f(t) = |t|$ on $[-\pi, \pi]$.

**(b)** Is there a term in the Fourier series for $g$ whose frequency is near the natural frequency of the system modeled by the DE? For the response found in Part (a), does this term have the largest amplitude?

**Problem 23.4.** Find the Fourier sine series for $f(x) = 1$ on $[0, \pi]$.

**Problem 24.5.** Solve $x' + kx = f(t)$, where $f(t)$ is the period $2\pi$ triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$. (You can use the known series for $f(t)$.)

**Extra problems if time.**

**Problem 24.6.** Solve $x'' + 4x = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$. Look out for resonance.

**Problem 23.7.** Find the Fourier series for the standard square wave shifted to the left so it's an even function, i.e., $sq(t + \pi/2)$.

**Problem 22.8.** **(a)** Compute the Fourier series for the even, period $2\pi$ function, with $f(t) = \pi t - t^2$ on $[0, \pi]$. The integral table provided should help.

**(b)** Carefully sketch the graph of the Fourier series.

**(c)** Challenge: Can you explain why the odd cosine coefficients are 0?

**Problem 22.9.** The function $f(t)$ has period $\pi$. Over the interval $0 \leq x < \pi$ we have $f(t) = \sin(t)$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$

**Problem 22.10.** Let $f(t)$ be the odd, period 2, amplitude 1 square wave. Carefully sketch the graph of the Fourier series.

**Problem 22.11.** Recall the Fourier series for the period $2\pi$ triangle wave $\text{tri}(t)$, where $\text{tri}(t) = |t|$ for $-\pi \leq t \leq \pi$:

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\,\text{odd}} \frac{\cos(nt)}{n^2}.$$

Set $t = 0$ and show $\sum_{n\,\text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$. (This is only for fun, we will not test on this sort of problem.)