ES.1803 Problem Section 12, Spring 2024 Solutions

Problem 25.1. *Consider the following heat equation with boundary conditions.*

PDE: $u_t(x,t) = 4u_{xx}(x,t)$, *for* $0 \le x \le \pi$, $0 \le t$.

BC: $u(0,t) = 0$, $u(\pi,t) = 0$.

(a) *Find the general solution.*

Solution: Step 1. Look for separated solutions to the PDE. That is, try a solution of the form $u(x,t) = X(x)T(t)$.

Substituting into the PDE gives

$$X(x)T'(t) = 4X''(x)T(t), \quad \text{a little algebra gives} \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{4T(t)} = \text{ constant } = -\lambda$$

(Since x and t are independent variables, when a function of x equals a function of t, both must be constant.)

A little more algebra gives two ordinary differential equations:

$$X'' + \lambda X = 0 \qquad T' + 4\lambda T = 0.$$

The equation for T has the solution $T(t) = ce^{-4\lambda t}$.

For X, the characteristic roots are $r = \pm\sqrt{-\lambda}$. There are 3 cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$, $T(t) = ce^{-4\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c$

Case (iii) $\lambda < 0$: Can ignore this case. It never produces nontrivial modal solutions. (For the record $X(x) = ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}$.)

Step 2. Modal solutions (separated solutions which also satisfy the BC)

For separated solutions, the BC are $X(0) = 0$, $X(\pi) = 0$.

(To see this: The BC $u(0,t) = X(0)T(t) = 0$ implies either $X(0) = 0$ or $T(t) = 0$. If $T(t) = 0$, then $u(x,t) = X(x)T(t) = 0$, i.e., u is the trivial solution. So, for nontrivial solutions, we must have $X(0) = 0$. Likewise, we need $X(\pi) = 0$.)

Case (i) $\lambda > 0$: BC: $X(0) = a = 0$, $X(\pi) = a\cos(\sqrt{\lambda}\,\pi) + b\sin(\sqrt{\lambda}\,\pi) = 0$.

Since $a = 0$, the second condition becomes $b\sin(\sqrt{\lambda}\,\pi) = 0$. Thus, either $b = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0$.

If $b = 0$, then $X(x) = 0$ and all we have found is the trivial solution.

If $\sin(\sqrt{\lambda}\,\pi) = 0$, then $\sqrt{\lambda} = n$ for some integer n.

So, for $\lambda = \sqrt{n}$, we have found some modal solutions.

$$X(x) = b\sin(nx) \quad \text{and} \quad T(t) = ce^{-4\lambda t} = ce^{-4n^2 t}.$$

Multiplying these together we get $u(x,t) = bc\sin(nx)e^{-4n^2 t}$.

There is no point in having both constants in the formula, so we drop the c. Also, to keep the solutions for different n separate, we add an index. Our modal solutions are

$$u_n(x,t) = b_n e^{-4n^2 t}\sin(nx) \quad \text{for } n = 1, 2, \ldots$$

Case (ii) $\lambda = 0$: BC: $X(0) = a = 0, \quad X(\pi) = a + b\pi = 0.$

The only solution to this is the trivial one $a = 0$, $b = 0$. So this case doesn't add any new modal solutions.

Case (iii) $\lambda < 0$: Ignore.

(You can easily check that this case does not produce any nontrivial solutions.)

Step 3: Using superposition, we get the general solution to the PDE satisfying BC:

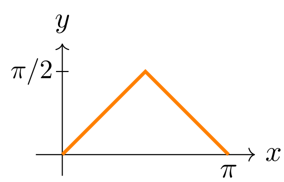
$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin(nx)$$

(b) *Now consider the initial condition (you should graph this).*

(IC) $u(x,0) = f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi/2 \\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$

Find the solution to the PDE that satisfies both the BC and the IC.

Solution: Here's the graph of $u(x,0)$ (the initial temperature distribution).



From the Part (a), we know the general solution to the PDE that satisfies the BC. As usual, the IC are used to determine the values of the coefficients.

Setting $t = 0$ in the general solution, we get:

$$u(x,0) = \sum b_n \sin(nx) = f(x) \quad \text{on } 0 \le x \le \pi$$

This is a sine series for $f(x)$. That is, b_n are the Fourier sine coefficients:

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

We can use the table or compute this integral by parts. (You should be able to do this!)

$$\begin{aligned}
b_n &= \frac{2}{\pi} \left(\int_0^{\pi/2} x \sin(nx)\, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin(nx)\, dx \right) \\
&= \frac{2}{\pi} \left(\left[-\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^{\pi/2} + \left[-\frac{\pi \cos(nx)}{n} + \frac{x \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_{\pi/2}^{\pi} \right) \\
&= \frac{2}{\pi} \left(-\frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} - \frac{\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n} + \frac{\pi \cos(n\pi/2)}{n} - \frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} \right) \\
&= \frac{4 \sin(n\pi/2)}{\pi n^2}
\end{aligned}$$

So, (since $\sin(n\pi/2) = 1, 0, -1, 0, 1$ ... for $n = 1, 2, 3, 4, 5, ...$)

$$\boxed{u(x,t) = \frac{4}{\pi}\left(e^{-4t}\sin(x) - \frac{e^{-36t}\sin(3x)}{9} + \frac{e^{-100t}\sin(5x)}{25} - \frac{e^{-196t}\sin(7x)}{49} + ...\right).}$$

(c) *If this models the temperature of a heated rod, what happens to the temperature over time? Which mode is the dominant mode?*

Solution: The temperature goes to 0 over the entire rod. The first mode is the dominant one, since it decays the slowest.

Heat equation applet: Take a look at the applet https://mathlets.org/mathlets/heat-equation/

Problem 26.2. **(a)** *Find the general solution to the following heat equation with inhomogeneous boundary conditions*

PDE: $u_t(x,t) = 4u_{xx}(x,t)$, *for* $0 \le x \le \pi$, $0 \le t$.

BC: $u(0,t) = 1$, $u(\pi,t) = 2$.

This has inhomogeneous boundary conditions. So we will use the strategy of finding a particular solution to the above and adding the general solution to the associated homogeneous equation. The homogeneous equation is a Topic 25 problem. Here is the solution:

$$u_h(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t}\sin(nx)$$

(If you haven't solved that problem yet, you should do that now.)

Solution: Our strategy is to find the general homogeneous solution by our previous methods and then guess at a particular solution to the inhomogeneous equation.

The homogeneous equation is

(H-PDE) $u_t(x,t) = 4u_{xx}(x,t)$, for $0 \le x \le \pi$, $0 \le t$.

(H-BC) $u(0,t) = 0$, $u(\pi,t) = 0$.

We gave the homogeneous solution above. If you haven't done this problem yet, you probably should try it now!

$$u_h(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t}\sin(nx)$$

For the particular solution we notice that the boundary conditions only depend on x, so we'll guess a steady state solution, i.e., one that doesn't change in time:

$$u_p(x,t) = X(x)$$

Substituting this into the PDE we get: $0 = 4X''(x)$, so $X(x) = a + bx$. Now matching the inhomogeneous BC we get

$$X(0) = a = 1 \text{ and } X(\pi) = a + b\pi = 2 \quad \Rightarrow a = 1,\ b = \frac{1}{\pi}.$$

Our finished solution to the problem is

$$u(x,t) = u_p(x,t) + u_h(x,t) = 1 + \frac{x}{\pi} + \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin(nx)$$

(b) *Find the solution that also satisfies the initial condition $u(x,0) = 2$.*

Solution: Using the solution from Part (a), we have

$$u(x,0) = 1 + x/\pi + \sum_{n=1}^{\infty} b_n \sin(nx) = 2$$

Rearranging terms we get: $\sum_{n=1}^{\infty} b_n \sin(nx) = 1 - x/\pi$. So the b_n are the Fourier sine coefficients for $1 - x/\pi$.

$$b_n = \frac{2}{\pi} \int_0^{\pi} (1 - x/\pi) \sin(nx)\, dx = \frac{2}{n\pi}$$

(We'll let you look up or compute the integrals.) Thus,

$$u(x,t) = 1 + \frac{x}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{-4n^2 t} \sin(nx)}{n}.$$

Problem 25.3. **(Linearity)** *Assume we have a heated rod of length L with its ends in ice baths. We can model this using the heat equation with boundary conditions.*

For functions $u = u(x,t)$, the PDE

$$\frac{\partial u}{\partial t}(x,t) = a \frac{\partial^2 u(x,t)}{\partial^2 x}$$

is the heat equation. In this problem we want to look at linearity of this equation and also of boundary conditions.

(a) *The PDE can be written as $\left(\frac{\partial}{\partial t} - a \frac{\partial^2}{\partial x^2} \right) u = 0$.*

We can use the language of operators: The partial differential operator $\mathcal{T} = \left(\frac{\partial}{\partial t} - a \frac{\partial^2}{\partial x^2} \right)$ is called the ***heat operator****. The heat equation is simply*

$$\mathcal{T} u = 0.$$

Show the heat operator is linear.

Solution: Remember that showing linearity is always easy –you just have to ask the question.

We need to show that

$$\mathcal{T}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{T} u_1 + c_2 \mathcal{T} u_2,$$

where c_1, c_2 are constants and u_1, u_2 are functions of (x,t). This follows easily from the linearity of (partial) derivatives:

$$\begin{aligned}\mathcal{T}(c_1u_1+c_2u_2) &= \frac{\partial(c_1u_1+c_2u_2)}{\partial t} - \frac{\partial^2(c_1u_1+c_2u_2)}{\partial x^2} \\ &= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - c_1\frac{\partial^2 u_1}{\partial x^2} - c_2\frac{\partial^2 u_2}{\partial x^2} \\ &= c_1\left(\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2}\right) + c_2\left(\frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2}\right) \\ &= c_1\mathcal{T}u_1 + c_2\mathcal{T}u_2. \qquad \text{QED}\end{aligned}$$

(b) *Show the heat equation $\mathcal{T}u = 0$ is homogeneous. That is, if u_1 and u_2 are solutions then so are $c_1u_1 + c_2u_2$.*

Solution: Again, all we have to do is ask the question. This follows by linearity

$$\mathcal{T}(c_1u_1+c_2u_2) = c_1\mathcal{T}u_1 + c_2\mathcal{T}u_2 = 0.$$

(c) *The boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ also have solutions, i.e., functions that satisfy the boundary conditions.*

Show the boundary conditions are linear and homogeneous. That is, we can superposition solutions and get solutions.

Solution: Assume u_1 and u_2 satisfy the boundary conditions, i.e.

$$u_1(0,t) = 0,\ u_1(L,t) = 0,\ u_2(0,t) = 0,\ u_2(L,t) = 0.$$

Let $u(x,t) = c_1u_1(x,t) + c_2u_2(x,t)$. Easily

$$u(0,t) = c_1u_1(0,t) + c_2u_2(0,t) = 0, \qquad \text{and } u(L,t) = c_1u(L,t) + c_2u_2(L,t) = 0.$$

This shows that the boundary conditions are linear and homogeneous.

(d) *Show that the combined system of the heat equation plus the given boundary conditions is linear and homogeneous.*

Solution: This is just about understanding what's being asked. The computations are trivial. A solution to the combined system is a function $u(x,t)$ satisfying

$$\mathcal{T}u = 0, \qquad u(0,t) = 0, \quad u(L,t) = 0.$$

Linear and homogeneous means that if u_1 and u_2 are solutions then so is $c_1u_1 + c_2u_2$ (for constants c_1, c_2). This follows directly from the previous parts of this problem.

Extra problems if time.

Problem 25.4. *(This problem uses a cosine series, so the $\lambda = 0$ case is important.)*

(a) *Solve the heat equation with insulated ends.*

PDE: $u_t = 3u_{xx}$ *for* $0 \le x \le 1$, $t > 0$.

BC: $u_x(0,t) = 0$, $u_x(1,t) = 0$

IC: $u(x,0) = x$.

Solution: Step 1: Separated solutions: try $u(x,t) = X(x)T(t)$.

Substitution gives

$$XT' = 3X''T \quad \Rightarrow \frac{X(x)''}{X(x)} = \frac{T(t)'}{3T(t)} = \text{constant} = -\lambda \quad \Rightarrow X'' + \lambda X = 0,\ T' + 3\lambda T = 0.$$

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = ce^{-3\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c$. So, $u(x,t) = (a+bx)c$,

Case (iii) $\lambda < 0$: Ignore, this case never produces nontrivial modal solutions.

Step 2: Modal solutions (separated solutions which also satisfy the BC)

For separated soltions, the BC are $X'(0) = 0$, $X'(1) = 0$.

Case (i) $\lambda > 0$: BC: $X'(0) = \sqrt{\lambda}b = 0$, $X'(1) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}) + \sqrt{\lambda}b\cos(\sqrt{\lambda})$.

The first condition gives $b = 0$. This implies $-a\sqrt{\lambda}\sin(\sqrt{\lambda}) = 0 \quad \Rightarrow a = 0$ or $\sin(\sqrt{\lambda}) = 0$.

We only get nontrivial solutions when $\sin(\sqrt{\lambda}) = 0$, i.e., when $\sqrt{\lambda} = n\pi$ for $n = 1, 2, \ldots$.

So $X(x) = a\cos(n\pi x)$, $T(t) = e^{-3n^2\pi^2 t}$ and we have found modal solutions:

$$\boxed{u_n(x,t) = a_n\cos(n\pi x)e^{-3n^2\pi^2 t} \quad \text{for } n = 1, 2, \ldots}$$

(We combined a and c into one constant and added the index n.)

Case (ii) $\lambda = 0$: BC: $X'(0) = b = 0$ and $X'(1) = b = 0$.

So, $b = 0$ and $a =$ anything, i.e., $X(x) = a$.

So we have one more modal solution. Let's call it $\boxed{u_0 = a_0/2}$.

Case (iii) $\lambda < 0$: Ignore, never produces nontrivial solutions.

Step 3: Superposition gives the general solution to PDE + BC.

$$u(x,t) = u_0(x,t) + \sum u_n(x,t) = \frac{a_0}{2} + \sum_n a_n\cos(n\pi x)e^{-3n^2\pi^2 t}$$

Step 4: Use the IC to determine the values of the coefficients.

IC: $u(x,0) = \dfrac{a_0}{2} + \sum_n a_n \cos(n\pi x) = x$.

This is the cosine series for the function $f(x) = x$. (Note, we were clever to arrange things so the constant term is $a_0/2$.) The cosine series for x is the same as the Fourier series for a scaled triangle wave. You can work out the scaling or just compute the integrals $a_n = 2\int_0^1 x\cos(n\pi x)\,dx$. Either way you get $a_0 = 1$, $a_n = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$

Thus,

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\,\text{odd}} \frac{\cos(n\pi x)\, e^{-3n^2\pi^2 t}}{n^2}.$$

(b) *Write out explicitly (compute values of coefficients) the first 4 nonzero terms when $t = 1/32$, i.e., write the first four terms of $u(x, 1/32)$. Use this to explain why, after a very short time, the constant and $n = 1$ term give a very good approximation of the solution.*

Solution: Using a calculator:

$$\begin{aligned} u(x, 1/32) &= \frac{1}{2} - \frac{4}{\pi^2}\left(e^{-3\pi^2/32}\cos(\pi x) + \frac{e^{-27\pi^2/32}}{9}\cos(3\pi x) + \frac{e^{-25\pi^2/32}}{25}\cos(5\pi x) + \ldots \right) \\ &= 0.5 - 0.505 * \cos(\pi x) - 3.42 \times 10^{-5}\cos(3\pi x) - 4.58 \times 10^{-12}\cos(5\pi x) - \ldots . \end{aligned}$$

The coefficients of the terms with $n = 3$ and higher are so small compared to the $n = 1$ term that, for $t > 1/32$, we have the excellent approximation

$$u(x,t) \approx 0.5 - \frac{4e^{-3\pi^2 t}}{\pi}\cos(\pi x).$$

Problem 25.5. *Solve the wave equation with boundary and initial equations.*

PDE: $y_{tt} = y_{xx}$ *for* $0 \le x \le 1$, $t > 0$.

BC: $y(0,t) = 0$, $y(1,t) = 0$

IC: $y(x,0) = 0$, $y_t(x,0) = 1$.

Solution: Step 1. Separated solutions: $y(x,t) = X(x)T(t)$.

Plug into the PDE: $XT'' = X''T \quad \Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{T(t)} = \text{constant} = -\lambda$.

This gives us two ordinary differential equations:

$$X'' + \lambda X = 0 \qquad T'' + \lambda T = 0.$$

For X, the characteristic roots are $r = \pm\sqrt{-\lambda}$. There are 3 cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = c\cos(\sqrt{\lambda}t) + d\sin(\sqrt{\lambda}t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c + dt$.

Case (iii) $\lambda < 0$: Ignore this case. (No nontrivial modal solutions.)

Step 2: Modal solutions (separated solutions which also satisfy the BC)

For separated solutions, the BC are $X(0) = 0$, $X(1) = 0$.

We look at each of the cases:

Case (i) $\lambda > 0$: BC: $X(0) = a = 0$,, $X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

Since $a = 0$, the second condition becomes $b\sin(\sqrt{\lambda}) = 0 \quad \Rightarrow b = 0$ or $\sin(\sqrt{\lambda}) = 0$.

If $b = 0$, then $X(x) = 0$ and all we have found is the trivial solution.

If $\sin(\sqrt{\lambda}) = 0$, then $\sqrt{\lambda} = n\pi$ for some integer n.

So, for $\lambda = \sqrt{n}\pi$, we have

$$X(x) = b\sin(n\pi x) \quad \text{and} \quad T(t) = c\cos(n\pi t) + d\sin(n\pi t).$$

Multiplying these together we have $y(x,t) = b\sin(n\pi x)(c\cos(n\pi t) + d\sin(n\pi t))$.

We drop the coefficient b (it's redundant) and index the modal solutions:

$$y_n(x,t) = \sin(n\pi x)(c_n\cos(n\pi t) + d_n\sin(n\pi t)) \quad \text{for } n = 1,\, 2, ...$$

Case (ii) $\lambda = 0$: BC: $X(0) = a = 0$, $X(1) = a + b = 0$.

The only solution is $a = 0$, $b = 0$. Thus, we have found only the trivial solution.

Case (iii) $\lambda < 0$: Ignore – only produces trivial modal solutions.

Step 3: Using superposition we, get the general solution to the PDE + BC:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)(c_n\cos(n\pi t) + d_n\sin(n\pi t))$$

Step 4: Use the initial conditions to determine the coefficients.

IC $y(x,0) = 0$: $y(x,0) = \sum c_n \sin(n\pi x) = 0$. This is a Fourier sine series for 0, i.e., all the coefficients $c_n = 0$.

IC $y_t(x,0) = 1$: $y_t(x,0) = \sum n\pi d_n \sin(n\pi x) = 1$. This is a Fourier sine series for 1 on [0,1]. We recognize this as the Fourier series for the odd period 2 square wave. So,

$$\sum n\pi d_n \sin(n\pi x) = 1 = \frac{4}{\pi}\sum_{n\, odd} \frac{\sin(n\pi x)}{n}.$$

That is, $n\pi d_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \quad \Rightarrow \; d_n = \begin{cases} \frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$

Our solution is

$$y(x,t) = \sum_{n=1}^{\infty} d_n \sin(n\pi x)\sin(n\pi t) = \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\sin(n\pi x)\sin(n\pi t)}{n^2}.$$