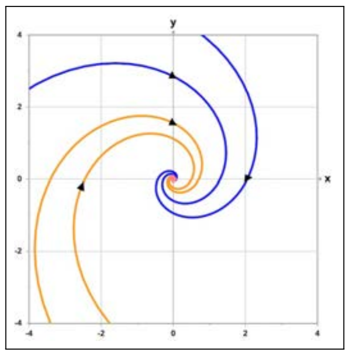
ES.1803 Problem Section 13, Spring 2024 Solutions

Problem 27.1. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$, *where* $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$. *What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. So the eigenvalues are $-1 \pm 2i$.

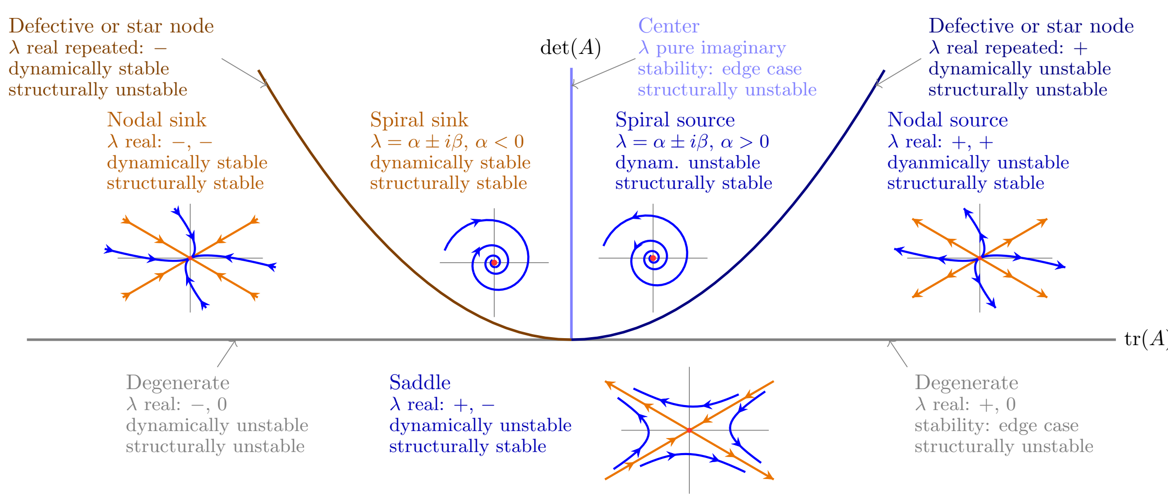
Thus the critical point at the origin is a spiral sink. Since the $2, 1$ entry of A is negative. The spiral turns in a clockwise manner. Spiral sinks are dynamically stable.

For spiral sinks, a qualitative phase portrait does not require computing the eigenvectors. By hand, we would just sketch clockwise spirals, spiraling in. (Of course, the graphing program we used here is more exact.)



Problem 27.2. *Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?*

Solution: Here is the diagram:



The open regions in the diagram all represent structurally stable systems. That is, nodal sources, spiral sources, nodal sinks, spiral sinks and saddles are all structurally stable. The lines represent structurally unstable systems, i.e., defective and star nodes, centers, degenerate systems.

(b) *Give the equation for the parabola in the diagram. Explain where is comes from.*

Solution: The characteristic equation is $\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0$. Therefore, the eigenvalues are

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}.$$

The parabola is the dividing line between real and imaginary root. That is it's where the discriminant (part under the square root) is 0. Its equation is

$$\mathrm{tr}(A)^2 - 4\det(A) = 0 \quad \Leftrightarrow \quad \det(A) = \frac{\mathrm{tr}(A)^2}{4}.$$

Problem 28.3. (a) *Sketch the phase portrait for* $x' = -x + xy, \quad y' = -2y + xy.$

Solution: First we find the critical points by factoring the equations:

$x' = x(-1+y) = 0 \Rightarrow x = 0$ or $y = 1$
$y' = y(-2+x) = 0 \Rightarrow x = 2$ or $y = 0$

So the only critical points are $(0,0)$ and $(2,1)$.

Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1+y & x \\ y & -2+x \end{bmatrix}$

At (0,0): $J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

This is the coefficient matrix A of our linearized system as (0,0). The eigenvalues are -1, -2, so this is a linearized nodal sink. Since nodal sinks are structurally stable, we also have a nonlinear nodal sink.

(As an aside, it is worth noting that the eigenvectors lie along the axes and clearly there are trajectories along each axis, i.e., if $y = 0$ the trajecory is along the x-axis.)

At (2,1): $J(2,1) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

This is the coefficient matrix A of our linearized system as (2,1). The eigenvalues are $\pm\sqrt{2}$, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

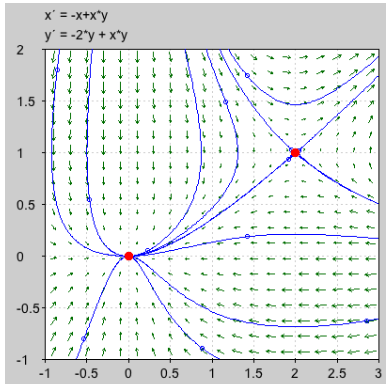
In order to sketch, we find the eigenvectors of the saddle:

The eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$,

$\lambda = \sqrt{2}$: $\quad A - \lambda I = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}$.

$\lambda = -\sqrt{2}$: $\quad A - \lambda I = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ -\sqrt{2} \end{bmatrix}$.

Now we can sketch the linearized systems near each critical point and tie them together.



(b) *Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.*

Solution: Alone each population has equation $x' = -x$ and $y' = -y$. So each would die off without the other. The interaction term xy is positive in both cases, so it seems these species cooperate to try to survive.

Unfortunately, it looks like there is a doomsday-extinction scenario. Depending on the initial conditions, Either the populations still die off to 0 (extinction) or else they explode to infinity (doomsday).

Problem 30.4. *The system for this equation is*

$$x' = 4x - x^2 - xy$$
$$y' = -y + xy$$

(a) *This models two populations with a predator-prey relationship. Which variable is the predator population?*

Solution: In the presence of y, the growth rate of x decreases. In the presence of x, the growth rate of y increases. Thus x is the prey population and y the predator population.

(b) *What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?*

Solution: Without prey, i.e., when $x = 0$, the DE for y is $y' = -y$. This is exponential decay. So eventually the predator population would go to 0.

Without predators, the equation for the prey becomes $x' = 4x - x^2$. This is the logistic equation with dynamically stable critical point $x = 4$ and dynamically unstable critical point $x = 0$. The prey population would eventially stabilize at 4.

(c) *There are three critical points. Find and classify them*

Solution: We can factor each of the equations to find the critical points:

$$x' = x(4 - x - y) = 0 \Rightarrow x = 0 \text{ or } 4 - x - y = 0$$
$$y' = y(-1 + x) \Rightarrow y = 0 \text{ or } x = 1.$$

The critical points are $(0,0), \quad (4,0), \quad (1,3)$.

The Jacobian is $J(x,y) = \begin{bmatrix} 4 - 2x - y & -x \\ y & -1 + x \end{bmatrix}$.

Considering each of the critical points in turn:

$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \lambda = 4, -1.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda = -4, 3.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(1,3) = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}.$

Characteristic equation: $\lambda^2 + \lambda + 3 = 0 \quad \Rightarrow \lambda = -1 \pm \sqrt{11}\, i.$

Complex eigenvalues with negative real part imply this is a linearized spiral sink. This is structurally stable, so the nonlinear critical point is also a spiral sink.

(d) *Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?*

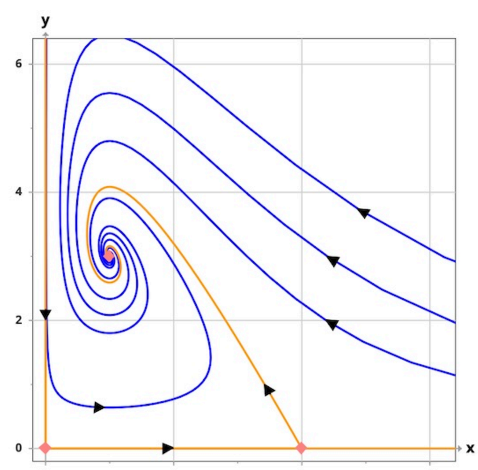
Solution: For the saddles, we need to find the eigenvectors. For the spiral, we need its direction.

$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \end{bmatrix}.$

The spiral at $(1,3)$ is counterclockwise because of the 3 in the lower left entry of $J(1,3)$.

Here is the phase portrait. Since we're talking about populations, the portrait only shows the first quadrant. All trajectories spiral into the critical point at (2,3). (Actually, there are a handful of trajectories along the axes that go asymptotically to the saddle points.)



Extra problems if time.

Problem 28.5. *Structural stability using the trace-determinant diagram: Will a non-structurally stable linearized critical point correctly predict the behavior of the nonlinear system at that point?*

Solution: Not necessarily. The linearized system is just an approximation of the nonlinear system. Non-structurally stable linearized systems might be qualitatively different from the nonlinear system they are approximating.

In the trace-determinant diagram the non-structurally stable systems are plotted on the boundary between different structurally stable systems.

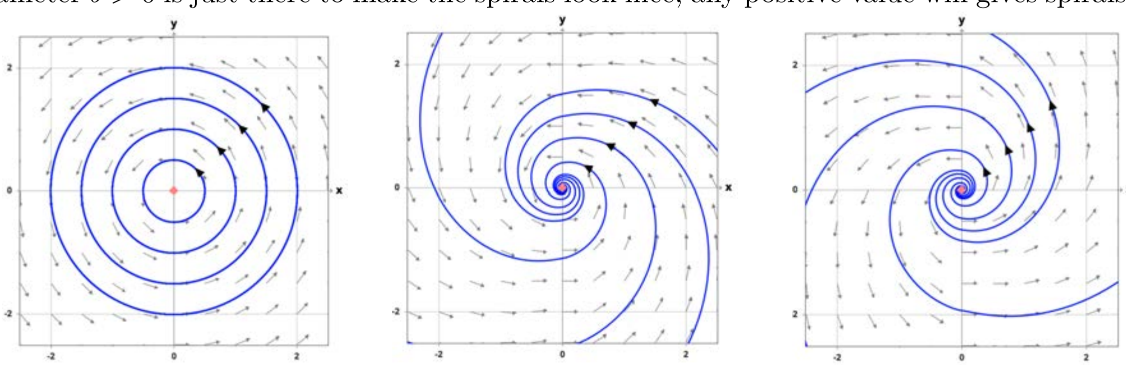
For example, a system with a linearized center at a critical point might be a nonlinear center or spiral. Here is a system which has a linearized center at the origin:

$$x' = -y + a \cdot x^3, \quad y' = x + a \cdot y \cdot x^2.$$

Clearly, for any value of a, the origin is a critical point. Also, for any a, $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
So the origin is a linearized center (turning counterclockwise).

If $a = 0$, the system is linear and the critical point is a genuine center. If $a < 0$ the origin is a nonlinear spiral sink. If $a > 0$ it is a nonlinear spiral source.

For drawing, we actually used the system: $x' = -y + ax \cdot |x|^b$, $y' = x + ay|x|^b$. The parameter $b > 0$ is just there to make the spirals look nice, any positive value will gives spirals.



Center: $a = 0$ Sink: $a = -0.4$, $b = 0.2$ Source: $a = 0.04$, $b = 0.2$

Problem 28.6. *Sketch the phase portrait for* $x' = x^2 - y$, $y' = x(1-y)$.
Draw one phase portrait for each possibility for the non-structurally stable critical point.

Solution: First we find the critical points.

Factoring the second equation: $y' = x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$.

Using these values in the second equation, $x' = x^2 - y = 0$ we find three critical points: $(0,0)$, $(1,1)$, $(-1,1)$.

Jacobian: $J(x,y) = \begin{bmatrix} 2x & -1 \\ 1-y & -x \end{bmatrix}$.

At $(0,0)$: $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues are $\pm i$, so this is a linearized center. The 1 in the lower left entry of the matrix implies it turns counterclockwise.

Since centers are not structurally stable, we can't be sure the nonlinear system has a center at $(0,0)$. It could be a center, spiral source or spiral sink. We sketch all three possibilities below.

At $(1,1)$: $J(1,1) = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$. The eigenvalues are 2, -1, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

In order to sketch, we find basic eigenvectors:

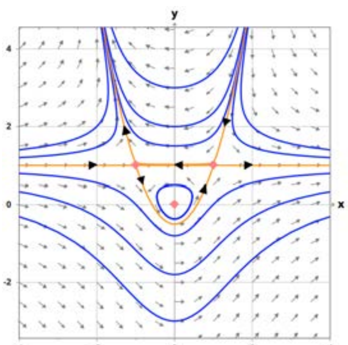
$\lambda = 2$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\quad \lambda = -1$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

At $(-1,1)$: $J(-1,1) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$. The eigenvalues are -2, 1, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

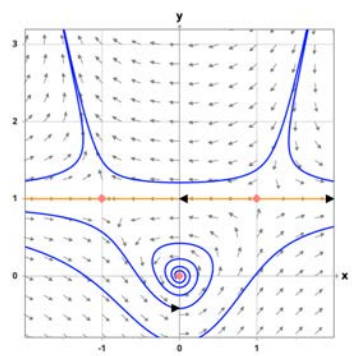
In order to sketch, we find basic eigenvectors:

$\lambda = -2$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\quad \lambda = 1$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

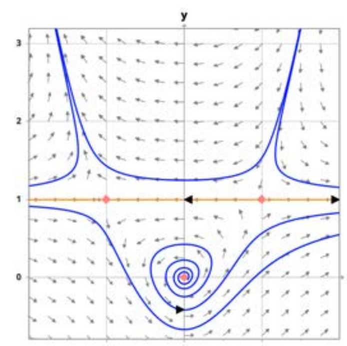
Here are sketches showing the three possible trajectories near the structurally unstable critical point.



Nonlinear center at (0,0)



Nonlinear spiral sink at (0,0)

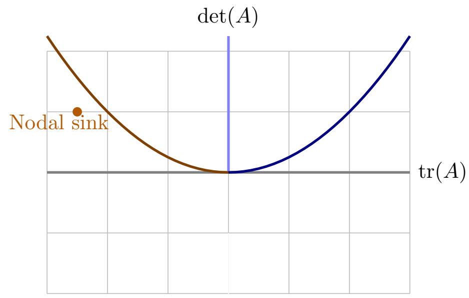


Nonlinear spiral source at (0,0)

Problem 27.7. *Consider the linear system* $\mathbf{x}' = A\mathbf{x}$.

(a) *Suppose* A *has* $\operatorname{tr}(A) = -2.5$ *and* $\det(A) = 1$. *Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?*

Solution: The diagram below shows the trace-determinant plane with the dividing lines included. The parabola has equation $\det(A) = \operatorname{tr}(A)^2/4$. The point $(-2.5, 1)$ is plotted. Since it is below the parabola in the third quadrant, it represents a nodal sink.



(b) *Compute the eigenvalues of this system and verify your answer in Part (a).*

Solution: The characteristic equation is

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 + 2.5\lambda + 1 = 0.$$

Therefore, the eigenvalues are $\dfrac{-2.5 \pm \sqrt{6.25-4}}{2} = -0.5, -2$. Since these are real and negative, the critical point at the origin is a nodal sink. This matches the answer in Part (a).

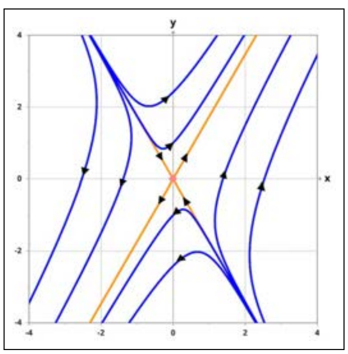
Problem 27.8. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$. *What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 - 2\lambda - 2 = 0$. So the eigenvalues are $1 \pm \sqrt{3}$.

Since the eigenvalues have opposite signs, the critical point at the origin is a saddle. Saddles are dynamically unstable.

For saddles, a qualitative phase portrait requires computing the eigenvectors. We find that an eigenvector corresponding to $\lambda - 1 + \sqrt{3}$ is $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, and one corresponding to $\lambda = 1 - \sqrt{3}$ is $\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

The modes give trajectories that are half lines. One mode has lines going out from the origin, and one has lines going in towards the origin. The mixed modal solutions are curves asymptotic to the modal trajectories at $t = \pm\infty$.



Problem 27.9. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. *What type of critical point is at the origin? Is it dynamically stable?*

Solution: First we find the eigenvalues. The characteristic equation is

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 + 4 = 0.$$

So the eigenvalues are $\pm 2i$. This means the critical point is a $\boxed{\text{center}}$.

The direction of rotation can be found by looking at the tangent vector at $(1, 0)$:

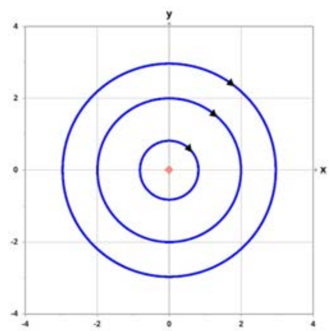
$$\mathbf{x}' = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

This tangent vector points down, which means that the ellipse is moving downwards at point $(1, 0)$ and so is moving clockwise.

Equivalently and more quickly: Because the $2, 1$ entry of A is negative, we know the trajectory turns in a $\boxed{\text{clockwise}}$ manner.

A center is on the boundary between dynamically asymptotically stable spiral sinks and dynamically unstable spiral sources. We call it an edge case. It is sometimes described as stable but not asymptotically stable.

For a center, when sketching a qualitative view of the phase portrait there is no need for eigenvectors. The trajectories are ellipses, which we have seen turn in a clockwise manner. For this system, the ellipses turn out to be perfect circles.

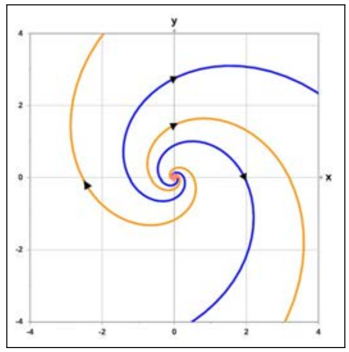


Problem 27.10. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. *What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$. So the eigenvalues are $1 \pm 2i$.

Thus the critical point at the origin is a spiral source. Since the $2, 1$ entry of A is negative. The spiral turns in a clockwise manner. Spiral sources are dynamically unstable.

For spiral sources, a qualitative phase portrait does not require computing the eigenvectors. By hand, we just sketch clockwise spirals, spiraling out.



Problem 28.11. *For the following system, draw the phase portrait by linearizing at the critical points.*

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Solution: First we find the critical points.

$$x' = 1 - y^2 = 0 \ \Rightarrow\ y = \pm 1$$
$$y' = x + 2y = 0 \ \Rightarrow\ x = -2y.$$

So the only critical points are $(-2, 1)$ and $(2, -1)$.

Jacobian: $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & -2y \\ 1 & 2 \end{bmatrix}$

At (-2,1): $J(-2, 1) = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$

This has eigenvalues are $1 \pm i$, so the critical point is is a linearized spiral source. The 1 in the lower left entry tells us it turns counterclockwise. Since spiral sources are structurally stable, we also have a nonlinear spiral source.

At (2,-1): $J(2, -1) = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$.

This has eigenvalues $1 \pm \sqrt{3}$, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

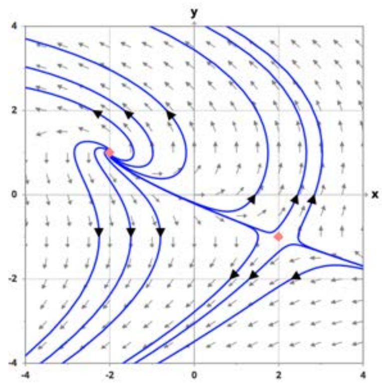
In order to sketch, we find the eigenvectors:

The eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$,

$\lambda = 1 + \sqrt{3}$: $\quad A - \lambda I = \begin{bmatrix} -1 - \sqrt{3} & 2 \\ 1 & 1 - \sqrt{3} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1 + \sqrt{3} \end{bmatrix}$.

$\lambda = 1 - \sqrt{3}$: $\quad A - \lambda I = \begin{bmatrix} -1 + \sqrt{3} & 2 \\ 1 & 1 + \sqrt{3} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1 - \sqrt{3} \end{bmatrix}$.

Now we can sketch the linearized systems near each critical point and tie them together.



Problem 28.12. *For the following system, draw the phase portrait by linearizing at the critical points.*

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Solution: First we find the critical points.

$$x' = x - y - x^2 + xy = 0$$
$$y' = -y - x^2 = 0.$$

The second equation implies $y = -x^2$. Putting this into the first equation gives

$$x + x^2 - x^2 - x^3 = x - x^3 = 0.$$

So, $x = 0,\ 1,\ -1$.

Thus the critical points are $(0,0)$, $(1,-1)$ and $(-1,-1)$.

Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1-2x+y & -1+x \\ -2x & 1 \end{bmatrix}$

At (0,0): $J(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

This has eigenvalues are ± 1, so the critical point is is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

In order to sketch, we find the eigenvectors. This is straighforward, eigenvectors for $\lambda = 1, -1$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively.

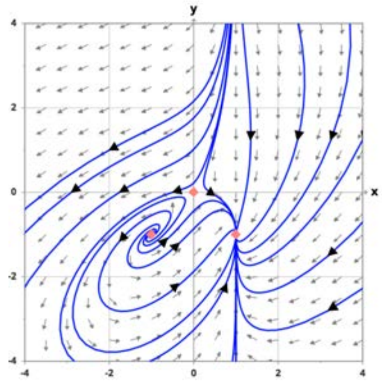
At (1,-1): $J(1,-1) = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix}$.

This has eigenvalues -2, -1, so this is a linearized nodal sink. Since nodal sinks are structurally stable, we also have a nonlinear nodal sink.

At (-1,-1): $J(-1,-1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$.

This has eigenvalues $(1 \pm \sqrt{7}\,i)2$, so this is a linearized spiral source. The 2 in the lower left entry of the Jacobian tells us the spiral is counterclockwise. Since spiral sources are structurally stable, we also have a nonlinear spiral source.

Now we can sketch the linearized systems near each critical point and tie them together.



Problem 30.13. *Consider the system of equations*

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) *Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?*

Solution: The Jacobian of the system is $J(x,y) = \begin{bmatrix} 39 - 6x - 3y & -3x \\ -4y & 28 - 2y - 4x \end{bmatrix}$.

(a) $J(0,0) = \begin{bmatrix} 39 & 0 \\ 0 & 28 \end{bmatrix}$. This is a diagonal matrix, so the eigenvalues are the diagonal entries: $\lambda = 39$, 28. Positive real eigenvalues imply the linearized critical point is a nodal source. This is structurally stable, so the nonlinear critical point is also a nodal source.

(b) *Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.*

Solution: $J(13,0) = \begin{bmatrix} -39 & -39 \\ 0 & -24 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -39, -24$. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(c) *Repeat Part (b) for the critical point (0,28).*

Solution: $J(0,28) = \begin{bmatrix} -45 & 0 \\ -112 & -28 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -45, -28$. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(d) *Repeat Part (b) for the critical point (5,8).*

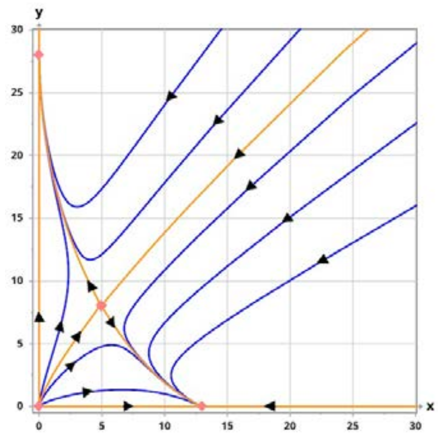
Solution: $J(5,8) = \begin{bmatrix} -15 & -15 \\ -32 & -8 \end{bmatrix}$. The characteristic equation is $\lambda^2 + 23\lambda - 360 = 0$. This has eigenvalues $\dfrac{-23 \pm \sqrt{23^2 + 4 \cdot 360}}{2}$. That is, it has one positive and one negative eigenvalue. Therefore, the linearized critical point is a saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

Note: We could also have identified this as a saddle because its determinant is negative.

(e) *Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?*

Solution: Here is the phase portrait. Note the separatix (in orange). It is made up of the trajectories that go asymptotically to the saddle point (5,8).

The relationship between the species is one of competition –you see that because both x' and y' have a $-xy$ term. In the long run one species dies out and the other stabilizes at the carrying capacity of the environment.



Extra problems if time.

Problem 30.14. *Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations*

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units $1/\text{years}$.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Solution: Original equations:

$$\begin{aligned} \text{sharks: } x' &= ax - pxy \\ \text{fish: } y' &= -by + qxy. \end{aligned}$$

The original equilibrium is (sharks, fish) $= (\frac{b}{q}, \frac{a}{p})$.

With warming:

$$\begin{aligned} x' &= (a - 0.1)x - pxy \\ y' &= -(b + 0.1)y + qxy \end{aligned}$$

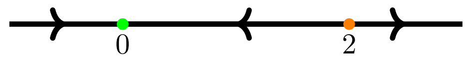
The new equilibrium is (sharks, fish) $= \left(\frac{b+0.1}{q}, \frac{a-0.1}{p}\right)$. So the equilibrium level of sharks increases. (And that of fish decreases.)

Problem 30.15. *The equations for this system are*

$$x' = x^2 - 2x - xy$$
$$y' = y^2 - 4y + xy$$

(a) *If this models two populations, what would happen to each of the populations in the absence of the other?*

Solution: If $y(t) = 0$, then $x' = x^2 - 2x$. This has critical points $x = 0, 2$ and phase line



So, without any predator ($y(t) = 0$), the prey population x will either crash to 0 or boom to infinity –at least according to this model.

The answer is the same for $y(t)$ if $x(t) = 0$.

(b) *There are four critical points. Find and classify them*

Solution: Again, we can factor to find the critical points.

$$x' = x(x - 2 - y) = 0 \Rightarrow x = 0 \text{ or } x - 2 - y = 0$$
$$y' = y(y - 4 + x) = 0 \Rightarrow y = 0 \text{ or } y - 4 - x = 0.$$

First let $x = 0$, then $y = 0$ or $y = 4$: two critical points (0,0), (0,4).

Next let $y = 0$, then $x = 0$ or $x = 2$: one more critical point (2,0).

Finally, solve $x - 2 - y = 0$, $y - 4 - x = 0$: one more critical (3,1).

The Jacobian is $J(x, y) = \begin{bmatrix} 2x - 2 - y & -x \\ y & 2y - 4 + x \end{bmatrix}$. Looking at each critical point in turn we get

$J(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \lambda = -2, -4$. Negative eigenvalues imply this is a linearized nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

$J(0, 4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow \lambda = -6, 4$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(2, 0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda = 2, -2$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(3, 1) = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - 4\lambda + 6 = 0 \Rightarrow \lambda = 2 \pm \sqrt{2}\, i$ Complex eigenvalues with positive real part imply this is a linearized spiral source. This is structurally stable, so the nonlinear critical point is also a spiral source.

(c) *Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?*

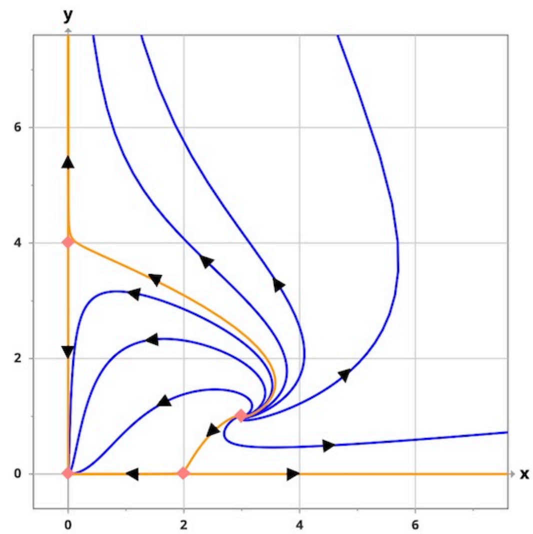
Solution: For the saddles, we need to find the eigenvectors. For the spiral, we need its direction.

$J(0,4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$J(2,0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The spiral at $(1,3)$ is counterclockwise because of the 1 in the lower left entry of $J(1,3)$.

Here is the phase portrait. The trajectories either go asymptotically to $(0,0)$ or to ∞. This looks like a predator-prey relationship. What seems more important, is that each population by itself is modeled by a doomsday-extinction equation. That is, either the population goes to ∞ or to 0. It's hard to tell exactly, but it seems that when the predator (y) goes to infinity, the prey (x) goes extinct.



Problem 28.16. *Consider the system:* $x' = x - 2y + 3, \quad y' = x - y + 2.$

(a) *Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?*

Solution: The equations for the critical points are

$$\begin{aligned} x' &= x - 2y + 3 = 0 \\ y' &= x - y + 2 = 0. \end{aligned}$$

This is a linear system of equations. The only solution is $(x,y) = (-1,1)$.

Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

So, $J(-1,1) = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$. Thus the linearized system at the critical point is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This has characteristic equation $\lambda^2 + 1 = 0$. So the eigenvalues are $\pm i$. This shows the linearized system is a center.

(b) *In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.*

Solution: This is an inhomogeneous linear system with constant input. So one way to make a phase portrait is to solve the equation and plot trajectories.

Since the input is constant, we guess a constant solution $\mathbf{x} = \mathbf{K}$. We find $\mathbf{x_p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (Not surprisingly this is the same as the critical point!)

The associated homogeneous system is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For the linear, homogeneous system, the coefficient matrix has eigenvalues $\pm i$. Thus the critical point at the origin is a center.

The general solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Since $\mathbf{x_p}$ is a constant, the general inhomogeneous solution is just the homogeneous solution translated by $(-1,1)$. This shows that the critical point at $(-1,1)$ is, indeed, a center.