ES.1803 Problem Section 13, Spring 2024

Problem 27.1. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.2. Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?

(b) Give the equation for the parabola in the diagram. Explain where is comes from.

Problem 28.3. **(a)** Sketch the phase portrait for $x' = -x + xy$, $y' = -2y + xy$.

(b) Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.

Problem 30.4. The system for this equation is

$$\begin{aligned} x' &= 4x - x^2 - xy \\ y' &= -y + xy \end{aligned}$$

(a) This models two populations with a predator-prey relationship. Which variable is the predator population?

(b) What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?

(c) There are three critical points. Find and classify them

(d) Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?

Extra problems if time.

Problem 28.5. Structural stability using the trace-determinant diagram: Will a non-structurally stable linearized critical point correctly predict the behavior of the nonlinear system at that point?

Problem 28.6. Sketch the phase portrait for $x' = x^2 - y$, $y' = x(1 - y)$.

Draw one phase portrait for each possibility for the non-structurally stable critical point.

Problem 27.7. Consider the linear system $\mathbf{x}' = A\mathbf{x}$.

(a) Suppose A has $\text{tr}(A) = -2.5$ and $\det(A) = 1$. Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?

(b) Compute the eigenvalues of this system and verify your answer in Part (a).

Problem 27.8. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.9. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.10. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 28.11. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Problem 28.12. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Problem 30.13. Consider the system of equations

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?

(b) Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.

(c) Repeat Part (b) for the critical point (0,28).

(d) Repeat Part (b) for the critical point (5,8).

(e) Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?

Extra problems if time.

Problem 30.14. Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units 1/years.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Problem 30.15. The equations for this system are

$$\begin{aligned} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{aligned}$$

(a) If this models two populations, what would happen to each of the populations in the absence of the other?

(b) There are four critical points. Find and classify them

(c) Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?

Problem 28.16. Consider the system: $x' = x - 2y + 3, \quad y' = x - y + 2.$

(a) Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?

(b) In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.