ES.1803 Problem Section 14, Spring 2024 Solutions

Topic 31 (nonlinear mechanical systems) is not officially part of the course, but these problems are fun and will give you more practice with nonlinear systems.

Problem 31.1. Nonlinear Spring

The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \quad \begin{cases} \textit{\textbf{hard if}}\ c < 0 & \textit{(cubic term adds to linear force)} \\ \textit{\textbf{soft if}}\ c > 0 & \textit{(cubic term opposes linear force).} \end{cases}$$

(a) *Convert this to a companion system of first-order equations.*

Solution: The companion system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m \end{aligned}$$

(b) *Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)*

Solution: Case 1. Hard spring ($c < 0$): One critical point at $(0, 0)$

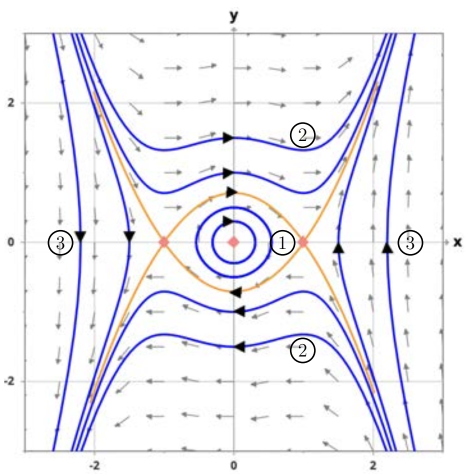
The Jacobian $J(x, y) = \begin{bmatrix} 0 & 1 \\ -k/m + 3cx^2/m & 0 \end{bmatrix}$

$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \Rightarrow \lambda = i\sqrt{k/m}$. So we have a linearized center. The problem statement tells us that this is also a nonlinear center.

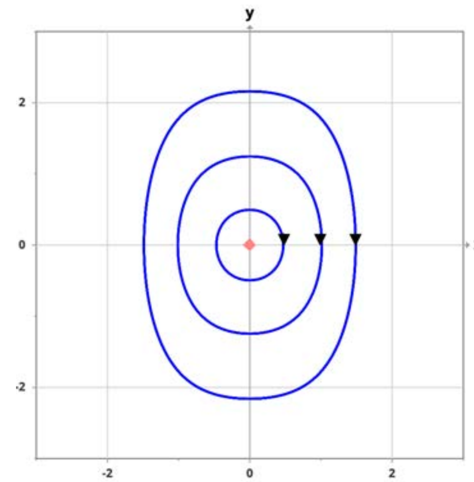
Case 2. Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: $J(0, 0)$ is the same as for the hard spring. This is a linearized center. The problem statement says it is also a nonlinear center.

$(\pm\sqrt{k/c}, 0)$: $J(\pm\sqrt{k/c}, 0) = \begin{bmatrix} 0 & 1 \\ 2k/m & 0 \end{bmatrix}$ (same for both). Thus we have linearized saddles and, by structural stability, nonlinear saddles. (You should find the eigenvectors to aid in sketching the phase portrait.)



Soft spring: $c > 0$



Hard spring: $c < 0$

(c) *(Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.*

Solution: We use a standard trick to get trajectories:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-kx + cx^3}{my}.$$

This is separable: $my\,dy = (-kx + cx^3)\,dy$. Integrating we get

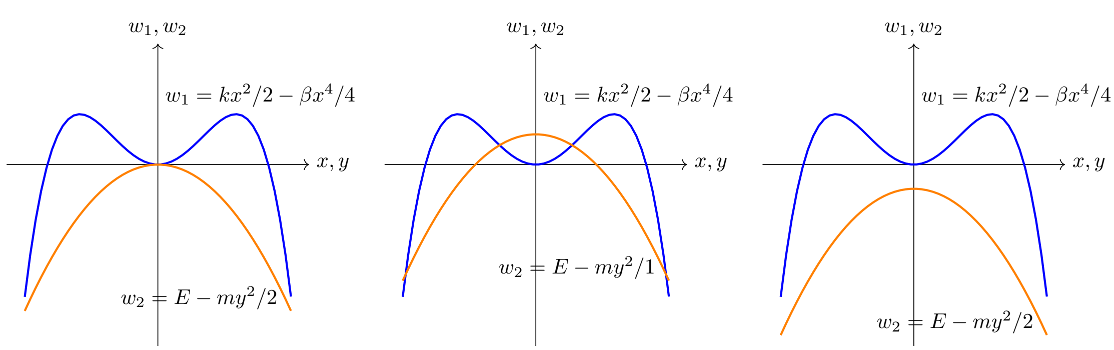
$$\underbrace{\frac{my^2}{2}}_{\text{kinetic energy}} + \underbrace{\frac{kx^2}{2} - \frac{cx^4}{4}}_{\text{potential energy}} = \underbrace{E}_{\text{total energy = constant}}.$$

If $c < 0$ (hard spring), then both energy terms on the right are positive, so x and y must be bounded. Then, for fixed x, there are at most two points on the trajectory. Thus we must have closed trajectories.

If $c > 0$ (soft spring), then, we can define w_1 and w_2 by

$$w_1(x) = \frac{kx^2}{2} - \frac{cx^4}{4}, \quad w_2(y) = E - \frac{my^2}{2}$$

Using $k > 0$, $m > 0$, we have the graphs of w_1, w_2 given below. Using the same graphical ideas as in the proof in the Topic 30 notes that the Volterra predator-prey equation has closed trajectories, this shows the phase plane for the soft spring is as shown above.



Plots of $w_1 = \frac{kx^2}{2} - \frac{cx^4}{4}$, $w_2 = E - y^2$

Different energy levels correspond to different types of trajectories. At the unstable equilibrium we compute $E = \frac{k^2}{4c}$. We have the following correspondence between energy level and trajector (using the labels on the soft-spring phase portrait above):

$E = 0$: Stable equilibrium.

$0 < E < \dfrac{k^2}{4c}$: Trajectories 1.

$E = \dfrac{k^2}{4c}$: Unstable equilibrium, or a trajectory going asymptotically to or from the unstable equilibrium.

$\dfrac{k^2}{4c} < E$: Trajectories 2.

$E < \dfrac{k^2}{4c}$ (including $E < 0$): Trajectories 3

Problem 31.2. *The* ***damped nonlinear spring*** *has equation*

$$m\ddot{x} = -kx + cx^3 - b\dot{x}.$$

(a) *Convert it to a system of first-order equations.*

(b) *Sketch a phase portrait for both the hard and soft springs.*

Solution: (a) The system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m - by/m \end{aligned}$$

(b) Hard spring ($c < 0$): One critical point at $(0, 0)$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2 - 4km}}{2m}$. So we have 3 possiblities:

(i) underdamped = linearized spiral sink;

(ii) overdamped = linearized nodal sink;

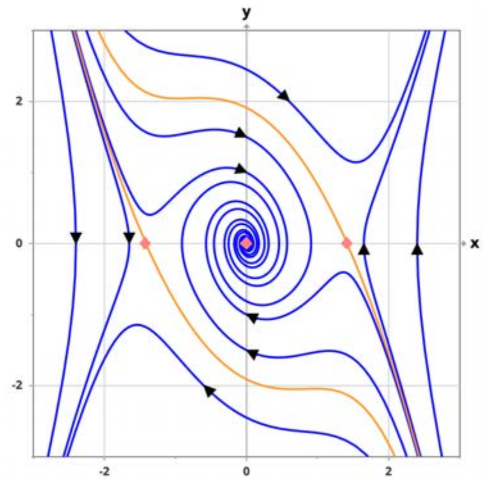
(iii) critically damped = defective sink.

In all cases we have a nonlinear sink. In case (iii), because it's not structurally stable, we would need to do more work to see what type of nonlinear sink we have.

Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: linearized sink (spiral, nodal or defective), so we have a nonlinear sink.

$(\pm\sqrt{k/c}, 0)$: linearized saddles, so we have nonlinear saddles.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-2
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2