ES.1803 Problem Section 2, Spring 2024 Solutions

Problem 3.1. *Solve the DE* $x' + 2x = f(t)$, $x(0) = 0$, *where* $f(t) = \begin{cases} 6 & \text{for } 0 \le t < 1 \\ 0 & \text{for } 1 \le t < 2 \\ 6 & \text{for } 2 \le t. \end{cases}$

Solution: First we solve the general cases (you can and should solve these by memory and inpsection).

IVP 1: $x' + 2x = 0,\ x(t_0) = b \quad \Rightarrow\ x(t) = be^{-2(t-t_0)}$.

IVP 2: $x' + 2x = 6,\ x(t_0) = b \Rightarrow x(t) = 3 + (b-3)e^{-2(t-t_0)}$.

For our problem:
Case $0 \le t < 1$: DE: $x' + 2x = 6,\quad x(0) = 0$.

So, using IVP 2, $x(t) = 3 - 3e^{-2t}$. For the next case: $x_1 = x(1) = 3(1 - e^{-2})$.

Case $1 \le t < 2$: DE: $x' + 2x = 0,\quad x(1) = x_1$.

So, using IVP 1, $x(t) = x_1 e^{-2(t-1)}$. For the next case: $x_2 = x(2) = x_1 e^{-2} = 3(e^{-2} - e^{-4})$.

Case $2 \le t$: DE: $x' + 2x = 6,\quad x(2) = x_2$.

So, using IVP 2, $x(t) = 3 + (x_2 - 3)e^{-2(t-2)}$.

Putting the cases together:

$$x(t) = \begin{cases} 3(1 - e^{-2t}) & \text{for } 0 \le t < 1 \\ x_1 e^{-2(t-1)} = 3(1 - e^{-2})e^{-2(t-1)} & \text{for } 1 \le t < 2 \\ 3 + (x_2 - 3)e^{-2(t-2)} = 3 + 3(-1 + e^{-2} - e^{-4})e^{-2(t-2)} & \text{for } 2 \le t. \end{cases}$$

Problem 4.2. (Polar coordinates)
Write $z = -1 + \sqrt{3}i$ *in polar form.*

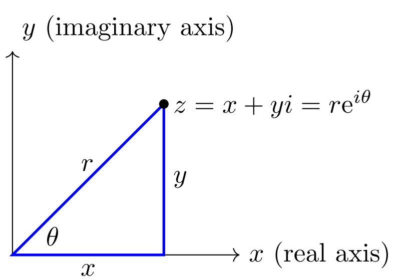
Solution: Easily:

$$|z| = 2, \quad \text{and} \quad \boxed{\text{Arg}(z) = \phi = \tan^{-1}(-\sqrt{3}/1) = 2\pi/3}.$$

(We know ϕ is in the 2nd quadrant.) So, $\boxed{z = 2e^{i\phi} = 2e^{i2\pi/3}.}$

Problem 4.3. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Solution:



Problem 4.4. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Solution: We start by writing -2 in polar form, being sure to include all values of the argument:

$$-2 = 2e^{i\pi + i2n\pi}.$$

Raising this to the power 1/5 gives

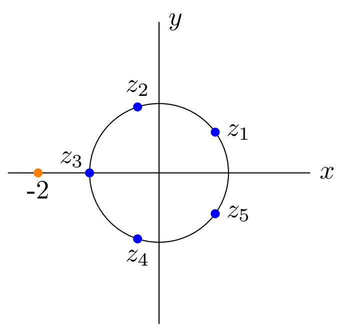
$$(-2)^{1/5} = 2^{1/5}e^{i\pi/5 + i2n\pi/5}.$$

Thus the 5 unique roots are:

$$z_1 = 2^{1/5}e^{i\pi/5}, \quad z_2 = 2^{1/5}e^{i3\pi/5}, \quad z_3 = 2^{1/5}e^{i5\pi/5}, \quad z_4 = 2^{1/5}e^{i7\pi/5}, \quad z_5 = 2^{1/5}e^{i9\pi/5}.$$

The only one of these that simplifies is $z_3 = 2^{1/5}e^{i5\pi/5} = -2^{1/5}$.

The figure below shows -2 and its fifth roots. Notice they are equally spaced around a circle of radius $2^{1/5}$.



Fifth roots of -2

Problem 4.5. (Complex replacement or complexification)
Compute $I = \int e^{4x}\cos(3x)\,dx$ *using complex techniques.*

Solution: Replacing $\cos(3x)$ by e^{i3x} we have: $I_c = \int e^{(4+3i)x}$, $I = \text{Re}(I_c)$.

Integrating: $I_c = \dfrac{e^{(4+3i)x}}{4+3i}$.

Polar form: $4 + 3i = 5e^{i\phi}$, where $\phi = \text{Arg}(4 + 3i) = \tan^{-1}(3/4)$ in Q1.

Thus, $I_c = \dfrac{e^{4x}}{5}e^{i(4x-\phi)}$. This implies

$$I = \text{Re}(I_c) = \frac{e^{4x}}{5}\cos(3x - \phi).$$

Problem 4.6. *Using the polar form, explain why* $|z^n| = |z|^n$ *and* $\arg(z^n) = n\arg(z)$ *for* n *a positive integer.*

Solution: In polar coordinates we have $z = re^{i\theta}$. So, $z^n = r^n e^{in\theta}$, i.e., $|z^n| = r^n = |z|^n$ and $\arg(z^n) = n\theta = n\arg(z)$. ■

Another way to say this is:

Magnitudes multiply, so $|z^n| = |z \cdot z \cdots z| = |z| \cdot |z| \cdots |z| = |z|^n$.

Arguments add, so $\arg(z^n) = \arg(z \cdot z \cdots z) = \arg(z) + \arg(z) + \cdots + \arg(z) = n \arg z$.

Problem 4.7. **(a)** *Write* $\cos(\pi t) - \sqrt{3}\sin(\pi t)$ *in the form* $A\cos(\omega t - \phi)$*.*

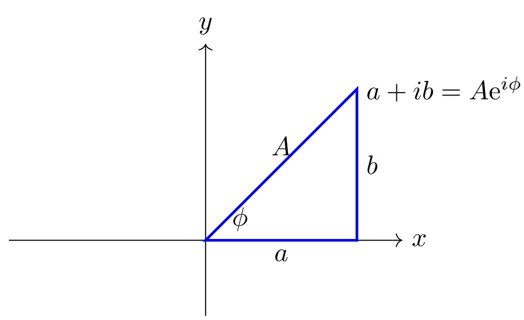
(b) *Write* $5\cos\left(3t + \frac{3\pi}{4}\right)$ *in the form* $a\cos(\omega t) + b\sin(\omega t)$*.*

(In each case, begin by drawing a right triangle with sides a and b, angle ϕ, hypotenuse A.)

Solution: This problem uses the identity

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi)$$

in which (A, ϕ) are the polar coordinates of the coefficients (a, b).



(a) We have the point $(a, b) = (1, -\sqrt{3})$ (in the 4th quadrant). So, $A = \sqrt{1+3} = 2$ and $\phi = \tan^{-1}(-\sqrt{3}) = -\pi/3$. Thus,

$$\cos(\pi t) - \sqrt{3}\sin(\pi t) = 2\cos\left(\pi t + \frac{\pi}{3}\right).$$

(b) We have $A = 5$ and $\phi = 3\pi/4$. So,

$$a = 5\cos\left(-\frac{3\pi}{4}\right) = -\frac{5}{\sqrt{2}}, \qquad b = 5\sin\left(-\frac{3\pi}{4}\right) = -\frac{5}{\sqrt{2}}.$$

Thus, $5\cos\left(3t + \dfrac{3\pi}{4}\right) = -\dfrac{5}{\sqrt{2}}\cos(3t) - \dfrac{5}{\sqrt{2}}\sin(3t)$.

Extra problems if time.

Problem 4.8. (Polar coordinates)
We know $-1 + \sqrt{3}\,i = 2e^{i2\pi/3}$. Use this to answer the following questions.

(a) *Compute the product* $(-1 + \sqrt{3}i)(a + bi)$ *(where a, b are real).*

Describe geometrically what multiplying by $-1+\sqrt{3}i$ does.

Solution: $(-1+\sqrt{3}i)(a+bi) = (-a-\sqrt{3}b) + (-b+\sqrt{3}a)i$.

In polar coordinates

$$(-1+\sqrt{3}i)re^{i\theta} = 2e^{2\pi/3}re^{i\theta} = 2re^{i(\theta+2\pi/3)}.$$

Multiplying $z = a+bi$ by this number multiplies the magnitude of z by 2, and increases the argument by $2\pi/3$, i.e., it expands by a factor of 2 and rotates by 120°.

(b) *What are the polar coordinates of $(-1+\sqrt{3}i)(a+bi)$ in terms of the polar coordinates of $a+bi = re^{i\theta}$?*

Solution: See answer to Part (a): The magnitude is $2r$ and the argument is $\theta + 2\pi/3$.

(c) *Describe the sequence of powers of $-1+\sqrt{3}i$, positive and negative.*

Solution: The powers of $-1+\sqrt{3}i$ spiral out, rotating counterclockwise by 120° each time and growing by a factor of 2. Successive negative powers rotate clockwise by 120° and shrink by a factor of $1/2$.

Problem 4.9. *Compute $\dfrac{1}{-2+3i}$ in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.*

Solution: In polar form $-2+3i = \sqrt{13}e^{i\theta}$, where $\theta = \arg(-2+3i) = \tan^{-1}(-3/2)$ in the second quadrant.

Therefore, $\dfrac{1}{-2+3i} = \dfrac{1}{\sqrt{13}e^{i\theta}} = \dfrac{1}{\sqrt{13}}e^{-i\theta}$.

Problem 4.10. *Make up and solve some simple algebra problems involving addition, subtraction, division, magnitude, complex conjugation.*

Solution: Provided by you!

Problem 4.11. *Write $3e^{i\pi/6}$ in rectangular coordinates.*

Solution: By Euler's formula: $3e^{i\pi/6} = 3\cos(\pi/6) + 3i\sin(\pi/6) = \boxed{3\sqrt{3}/2 + i3/2.}$

Problem 4.12. *Find a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.*

Solution: First note, $\cos(3\theta) = \text{Re}(e^{3i\theta})$. We know,

$$e^{3i\theta} = (\cos(\theta) + i\sin(\theta))^3 = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Taking the real part, we have $\boxed{\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta).}$

Problem 4.13. *The point of this problem is to help you distinguish between taking the real part of a function and finding which members of a* family of functions *are real-valued.*

(a) *Show the inverse Euler formulas are true:*

$$\cos(t) = (e^{it} + e^{-it})/2, \qquad \sin(t) = (e^{it} - e^{-it})/2i.$$

Solution: Use Euler's formula:

$$e^{it} = \cos(t) + i\sin(t)$$
$$e^{-it} = \cos(t) - i\sin(t)$$

Adding these two formulas gives $e^{it} + e^{-it} = 2\cos(t)$. Dividing by 2 then gives the inverse Euler formula for $\cos(t)$.

Likewise, subtracting the two formulas gives $e^{it} - e^{-it} = 2i\sin(t)$. Now, dividing by $2i$ gives the formula for $\sin(t)$.

(b) *Find all the real-valued functions of the form $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are complex constants.*

(ii) Using Euler's formula we know that

$$\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it} = (\tilde{c}_1 + \tilde{c}_2)\cos(t) + i(\tilde{c}_1 - \tilde{c}_2)\sin(t)$$

If this is real-valued then the coefficients of $\cos(t)$ and $\sin(t)$ must be real:

$\tilde{c}_1 + \tilde{c}_2$ real implies $\mathrm{Im}(\tilde{c}_2) = -\mathrm{Im}(\tilde{c}_2)$.

$i(\tilde{c}_1 - \tilde{c}_2)$ real implies $\mathrm{Re}(\tilde{c}_2) = \mathrm{Re}(\tilde{c}_2)$.

Thus $\tilde{c}_1$ and $\tilde{c}_2$ are complex conjugates, say $\tilde{c}_1 = a - ib$ and $\tilde{c}_2 = a + ib$. Then

$$\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it} = 2a\cos(t) + 2b\sin(t)$$

Changing notation slightly, the answer is $\boxed{x(t) = a\cos(t) + b\sin(t).}$

Problem 4.14. *Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.*

Solution: Let $\tilde{c} = a + ib$. Expanding x we get

$$x(t) = e^{2t}(a + ib)(\cos(3t) + i\sin(3t)) = e^{2t}(a\cos(3t) - b\sin(3t) + i(a\sin(3t) + b\cos(3t)))$$

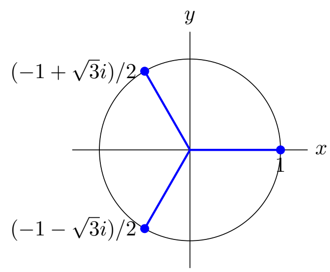
It's clear that the imaginary part can only be 0 if $a = b = 0$. So the only such real-valued function is $x(t) = 0$.

Problem 4.15. *Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.*

Solution: We know one cube root is 1. This is on the unit circle and the three roots are evenly spaced around the circle. So the other two are at $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Since $2\pi/3 = 120^\circ$ and $4\pi/3 = 240^\circ$, we can use our knowledge of 30, 60, 90 triangles to write the roots as

$$1, \qquad e^{2\pi i/3} = \frac{-1 + \sqrt{3}i}{2}, \qquad e^{4\pi i/3} = \frac{-1 - \sqrt{3}i}{2}$$

The figure below shows the three cube roots of 1.



Cube roots of 1

Problem 4.16. *Express in the form $a + bi$ the 6 sixth roots of 1.*

Solution: In polar form $1 = e^{i2\pi k}$, so

$$\begin{aligned} 1^{1/6} &= e^{i2\pi k/6} = e^{i\cdot 0},\ e^{i\pi/3},\ e^{i2\pi/3},\ e^{i3\pi/3},\ e^{i5\pi/3},\ e^{i5\pi/3} \\ &= 1,\ \frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -1,\ -\frac{1}{2} - i\frac{\sqrt{3}}{2},\ \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$

Problem 4.17. *Use Euler's formula to derive the trig addition formulas for* sin *and* cos*.*

Solution: Use $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$.

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \\ e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \end{aligned}$$

Equating the two expressions above, we have:

$$(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Equating the real and imaginary parts, we get the trig addition formulas:

$$\begin{aligned} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) &= \cos(\alpha+\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) &= \sin(\alpha+\beta). \end{aligned}$$

Problem 4.18. *Suppose $z^n = 1$. What must $|z|$ be? What are the possible values of $\arg(z)$, if $z^n = 1$?*

Solution: $|z|^n = 1$, and $|z| > 0$, so $|z|$ must be 1.

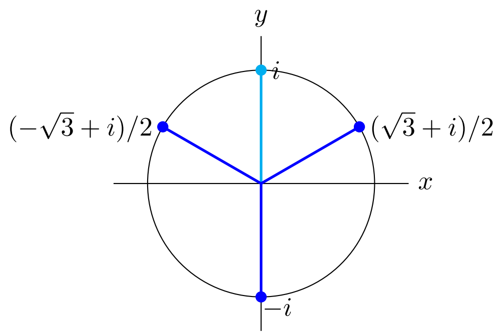
We must have $n\arg(z)$ is a multiple of 2π. So, $\arg(z) = 2m\pi/n$ for some integer n.

Problem 4.19. *Find the cube roots of i.*

Solution: We know that $i = e^{i\pi/2+2m\pi i}$, so the third roots are of the form $e^{i\pi/6+2m\pi i/3}$. The three unique roots are

$$e^{i\pi/6} = (\sqrt{3}+i)/2, \quad e^{i5\pi/6} = (-\sqrt{3}+i)/2, \quad e^{i9\pi/6} = -i.$$

The figure below shows the three cube roots of i.



Cube roots of i

Problem 4.20. *By using $(e^{it})^4 = e^{4it}$ and Euler's formula, find an expression for $\sin(4t)$ in terms of powers of $\cos(t)$ and $\sin(t)$,*

Solution: Compute

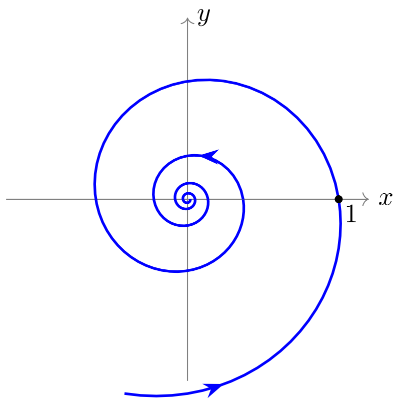
$$\begin{aligned} e^{4it} &= (e^{it})^4 \\ &= (\cos(t) + i\sin(t))^4 \\ &= \cos^4(t) + 4i\cos^3(t)\sin(t) - 6\cos^2(t)\sin^2(t) - 4i\cos(t)\sin^3(t) + \sin^4(t) \\ &= \left(\cos^4(t) - 6\cos^2(t)\sin^2(t) + \sin^4(t)\right) + i\,\left(4\cos^3(t)\sin(t) - 4\cos(t)\sin^3(t)\right) \end{aligned}$$

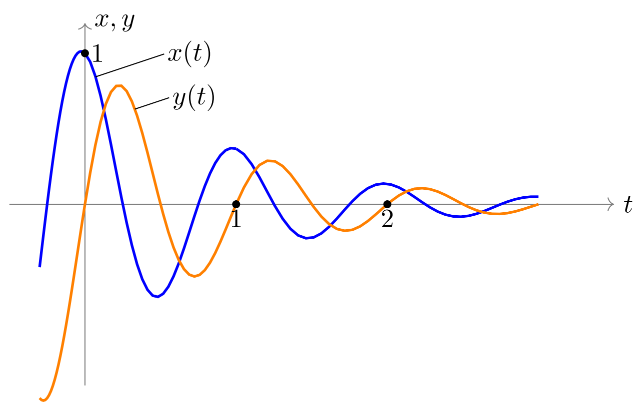
So, $\sin(4t) = \text{Im}(e^{4it}) = 4\cos^3(t)\sin(t) - 4\cos(t)\sin^3(t)$.

Problem 4.21. *Trajectories of $e^{(a+bi)t}$ can vary a lot, depending upon the value of the complex number $a+bi$. The "Complex Exponential" Mathlet shows this clearly. Invoke this applet if you can: `https://mathlets.org/mathlets/complex-exponential/`. You can use it to gain insight into the following questions.*

(a) *Sketch the trajectory of the complex-valued function $e^{(-1+2\pi i)t}$, and the graphs of its real and imaginary parts.*

Solution: This is a spiral moving towards the origin and turning counterclockwise. The real part is $e^{-t}\cos(2\pi t)$: a "damped sinusoid" with value 1 at $t = 0$. The imaginary part is $e^{-t}\sin(2\pi t)$: a damped sinusoid with value 0 at $t = 0$ and positive derivative there.





Left: Spiral in to origin. Right: Graphs of $x(t)$, $y(t)$.

(b) *For each of the following shapes, decide on all the values of $a+bi$ for which the trajectory of $e^{(a+bi)t}$ has this shape.*

(i) A circle centered at 0, traversed counterclockwise. What circles are possible?

(ii) A circle centered at 0, traversed clockwise.

(iii) A ray (straight half line) heading away from the origin.

(iv) A curve heading to zero as $t \to \infty$.

Solution: This will all depend upon Euler's formula

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$$

Notice that $|e^{(a+bi)t}| = e^{at}$ and $\text{Arg}(e^{(a+bi)t}) = bt$.

(i) This can only happen if the magnitude is constant: so $a = 0$. To go counterclockwise, we must have $b > 0$. Ans: bi, $b > 0$: the "positive imaginary axis." The circle must be the unit circle.

(ii) Again $a = 0$, but now $b < 0$: the "negative imaginary axis."

(iii) Now b must be zero. For the magnitude to be increasing, we must have $a > 0$. Answer: real a, $a > 0$: the postive real axis.

(iv) For this we must have $a < 0$. b can be anything. So: $a + bi$ with $a < 0$: the left half plane.

Problem 4.22. *Write $\cos(2t) + \sin(2t)$ in the form $A\cos(\omega t - \phi)$.*

Solution: The coefficients are $(a, b) = (1, 1)$, which have polar coordinates $A = \sqrt{2}$, $\phi = \frac{\pi}{4}$. So, $\cos(2t) + \sin(2t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$.