ES.1803 Problem Section 4, Spring 2024 Solutions

Problem 6.1. *Let $P(D) = D^2 + 6D + 5I$. Find the general real-valued solution to each of the following.*

(a) $P(D)x = e^{-2t}$.

Solution: The characteristic polynomial is $P(r) = r^2 + 6r + 5$. Using the exponential response formula (ERF) to find a particular solution.

$P(-2) = -3$, so $\boxed{x_p(t) = \frac{e^{-2t}}{P(2)} = -\frac{e^{-2t}}{3}.}$

The characteristic roots are $-1, -5$, so the general homogeneous solution is

$$\boxed{x_h(t) = c_1 e^{-t} + c_2 e^{-5t}}.$$

By superposition, the general solution to the DE is $\boxed{x(t) = x_p(t) + x_h(t) = -\frac{e^{-2t}}{3} + c_1 e^{-t} + c_2 e^{-5t}.}$

(b) $P(D)x = \cos(3t)$.

Solution: We'll use the sinusoidal response formula (SRF) to find a particular solution $x_p(t)$.

$P(3i) = -4 + 18i$. So, $|P(3i)| = 2\sqrt{85}$ and $\boxed{\phi = \text{Arg}(P(3i)) = \tan^{-1}(-9/2) \text{ in Q2}}$.

Thus the SRF gives $\boxed{x_p(t) = \frac{\cos(3t - \phi)}{|P(3i)|} = \frac{\cos(3t - \phi)}{2\sqrt{85}}.}$

Using the homogeneous solution from Part (a), the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

Alternatively, we could have used complex replacement with the DE $P(D)z = e^{3it}$ –see the answer to Part (c)

(c) $P(D)x = e^{2t}\cos(3t)$.

Solution: We start by using complex replacement to get the equation:

$$P(D)z = e^{(2+3i)t}, \text{ with } x = \text{Re}(z).$$

The ERF gives a particular solution $z_p(t) = \frac{e^{(2+3i)t}}{P(2+3i)}$.

Computing: $P(2+3i) = 12+30i$, so $|P(2+3i)| = 6\sqrt{29}$ and $\boxed{\phi = \text{Arg}(P(2+3i)) = \tan^{-1}(5/2) \text{ in Q1}.}$ Therefore,

$$z_p(t) = \frac{e^{2t}e^{i(3t-\phi)}}{6\sqrt{29}}, \text{ and } \boxed{x_p(t) = \text{Re}(z_p(t)) = \frac{e^{2t}}{6\sqrt{29}}\cos(3t - \phi).}$$

Again, using the homogeneous solution from Part (a), we have the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

(d) $P(D)x = e^{-t}$.

Solution: Since $P(-1) = 0$, we use the extended ERF: $\boxed{x_p(t) = te^{-t}/P'(-1) = te^{-t}/4.}$

Again, using the homogeneous solution from Part (a), we have the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

Problem 6.2. **(a)** *Solve $x'' + 4x = \cos(\omega t)$ for all possible values of ω.*

Solution: We know that $P(i\omega) = 4 - \omega^2$. In polar form we have

$$P(i\omega) = \begin{cases} |4-\omega^2| & \text{if } 0 < \omega < 2 \\ 0 & \text{if } \omega = 2 \\ |4-\omega^2|e^{i\pi} & \text{if } \omega > 2 \end{cases}$$

So, for $\omega \neq 2$, the SRF gives a particular solution

$$x_p(t) = \begin{cases} \frac{\cos(\omega t)}{|4-\omega^2|} & \text{if } 0 < \omega < 2 \\ \frac{\cos(\omega t - \pi)}{|4-\omega^2|} = -\frac{\cos(\omega t)}{|4-\omega^2|} & \text{if } \omega > 2 \end{cases}$$

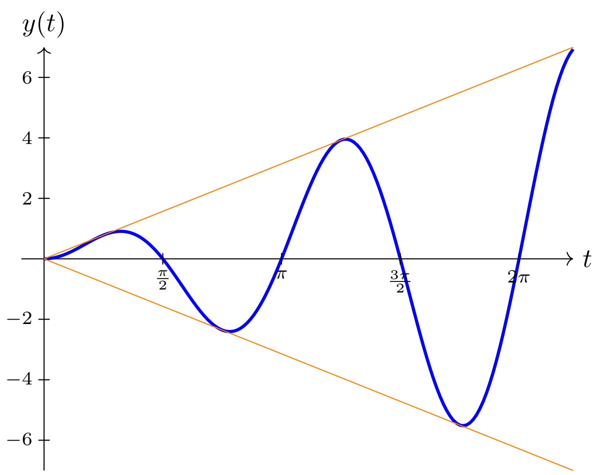
For $\omega = 2$ we need the extended SRF. First we compute $P'(2i) = 4i = 4e^{\pi i/2}$. So,

$$\boxed{x_p(t) = \frac{t\cos(2t - \pi/2)}{4} = \frac{t\sin(2t)}{4}.}$$

The homogeneous solution is $\boxed{x_h(t) = c_1\cos(2t) + c_2\sin(2t).}$ As always, the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

(b) *Plot the graph of your particular solution for $\omega = 2$.*

Solution: We have $x_p(t) = t\sin(2t)/4$.



Resonant response

Problem 7.3. *Solve $x' + 3x = t^2 + t$.*

Solution: Guess a trial solution of the form $x_p(t) = At^2 + Bt + C$ (same degree as the input). Substitute the guess into the DE (we don't show the algebra):

$$x_p' + 3x_p = 3At^2 + (2A + 3B)t + (B + 3C) = t^2 + t.$$

Equate the coefficents of the polynonials on both sides of the equation:

$$\begin{array}{llllll} \text{Coeff. of } t^2\text{:} & 3A & & & & = 1 \\ \text{Coeff. of } t\text{:} & 2A & + & 3B & & = 1 \\ \text{Coeff. of } 1\text{:} & & + & B & +3C & = 0 \end{array}$$

This triangular system is easy to solve: $A = 1/3$, $B = 1/9$, $C = -1/27$. Therefore, a particular solution is

$$\boxed{x_p(t) = \frac{1}{3}t^2 + \frac{1}{9}t - \frac{1}{27}.}$$

The homogeneous solution is $\boxed{x_h(t) = Ce^{-3t}.}$

The general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

Problem 8.4. *Is the system $x'' + x' + 4x = 0$ stable?*

Solution: Short answer: second-order with positive coefficients implies $\boxed{\textit{stable}}$.

Longer answer: characteristic roots are $r = \frac{-1\pm\sqrt{1-16}}{2}$. Since both roots have a negative real part, the system is stable.

Problem 8.5. *Is a 4th order system with roots $\pm 1, -2 \pm 3i$ stable. Which solutions to the homogeneous DE go to 0 as $t \to \infty$?*

Solution: No, the root $r = 1$ is positive so the system is not stable.

The general homogeneous solution is

$$x_h(t) = c_1e^t + c_2e^{-t} + c_3e^{-2t}\cos(3t) + c_4e^{-2t}\sin(3t),$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

The solutions that go to 0 are the ones with $c_1 = 0$, i.e., those of the form

$$x(t) = c_2e^{-t} + c_3e^{-2t}\cos(3t) + c_4e^{-2t}\sin(3t),$$

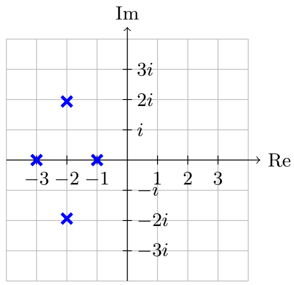
where c_2, c_3, c_4 are arbitrary constants.

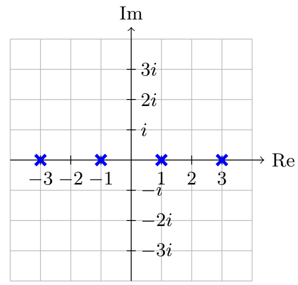
Problem 8.6. *For what k is the system $x' + kx = 0$ stable?*

Solution: Since the characteristic root is $r = -k$, this is stable when $k > 0$.

A better way to see this is, if $k > 0$ the system is one of exponential decay. If $k < 0$ it is one of exponential growth. If $k = 0$ it is an edge case. Some people will say it's stable but not asymptotically stable.

Problem 8.7. **(a)** *The pole diagram below on the left shows the characteristic roots of the system $P(D)x = 0$.*





Left: pole diagram for Part (a). *Right: diagram for Part (b)*

(i) What is the order of the system?

(ii) Is the system stable?

(iii) Is the system oscillatory?

(iv) What is the exponential decay rate for the general solution?

Solution: (i) There are 4 roots, so the order of the system is 4.

(ii) All the roots have negative real part, so the system is stable.

(iii) Since some of the roots are complex, the system is oscillatory.

(iv) The root with the least negative real part, i.e., the right-most root, controls the decay rate. The general solution decays like e^{-t}.

(b) *Repeat Part (a) for the pole diagram on the right.*

Solution: (i) There are 4 roots, so the order of the system is 4.

(ii) Some roots have positive real part, so the system is unstable.

(iii) All the roots are real, so the system is not oscillatory.

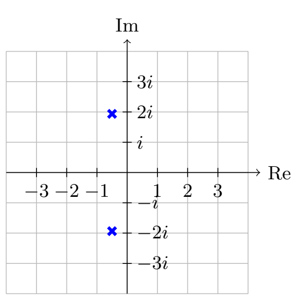
(iv) The general system does not decay, it grows like e^{3t}. We could make a case for saying the system decays like e^{3t}.

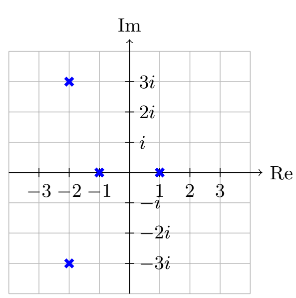
Problem 8.8. *Consider the following systems.*

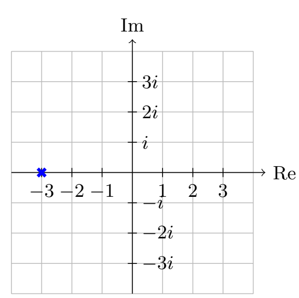
(i) $x'' + x' + 4x = 0$
(ii) A fourth-order system with roots ± 1, $-2 \pm 3i$
(iii) $x' + 3x = 0$.

Draw the pole diagram for each of these systems and say how it relates to the stability of the system.

Solution: The pole diagrams are shown in order for (i), (ii) and (iii).







Stability requires all the roots have negative real parts. That is, all the poles are in the left half plane. We see that (i) and (iii) are stable, but (ii) is not.

Extra problems if time.

Problem 6.9. (Sinusoidal response formula (SRF))
Let $P(D) = D^2 + 4D + 3$. Find a solution to $P(D)x = \cos(2t)$

Solution: The SRF says a particular solution is given by

$$x_p(t) = \frac{\cos(2t - \phi)}{|P(2i)|}, \text{ where } \phi = \operatorname{Arg}(P(2i)).$$

Computing: $P(2i) = -1 + 8i$, so $|P(2i)| = \sqrt{65}$ and $\boxed{\phi = \operatorname{Arg}(P(2i)) = \tan^{-1}(-8) \text{ in Q2}}$.

Thus, $\boxed{x_p(t) = \dfrac{\cos(2t - \phi)}{\sqrt{65}}}$.

Problem 6.10. *Solve $P(D)x = x'' + 4x' + 5x = e^{-t}\cos 2t$.*

Do this using complex replacement. Give the general solution.

Solution: Particular solution:
Complex replacement:

$$P(D)z = e^{-t}e^{2ti} = e^{(-1+2i)t} \qquad (x = \operatorname{Re}(z))$$

ERF: $z_p(t) = \dfrac{e^{(-1+2i)t}}{P(-1+2i)} = \dfrac{e^{(-1+2i)t}}{-2+4i}$.

Side work: $-2 + 4i = 2\sqrt{5}e^{i\phi}$, where $\tan\phi = -2$, ϕ in second quadrant.

So, $z_p(t) = \frac{e^{-t}}{2\sqrt{5}}e^{(2t-\phi)i}$. Taking the real part, $x_p(t) = \operatorname{Re}(z_p) = \frac{e^{-t}}{2\sqrt{5}}\cos(2t - \phi)$.

Homogeneous solution: $(P(D)x = 0)$

Characteristic equation: $r^2 + 4r + 5 = 0 \Rightarrow r = -2 \pm i$.

So the general homogeneous solution is $x_h(h) = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$.

General solution to $P(D)x = e^{-t}\cos 2t$: $\boxed{x(t) = x_p + x_h = \dfrac{e^{-t}}{2\sqrt{5}}\cos(2t - \phi) + c_1e^{-2t}\cos t + c_2e^{-2t}\sin t.}$

Problem 6.11. **(a)** *Show directly from the definition that $P(D) = D^3 + 6D^2 + 7I$ is a linear operator.*

Solution: We have to apply $P(D)$ to a linear combination of functions and see that it behaves properly, i.e., that for functions x_1, x_2 and constants c_1, c_2

$$P(D)(c_1x_1 + c_2x_2) = c_1P(D)x_1 + c_2P(D)x_2.$$

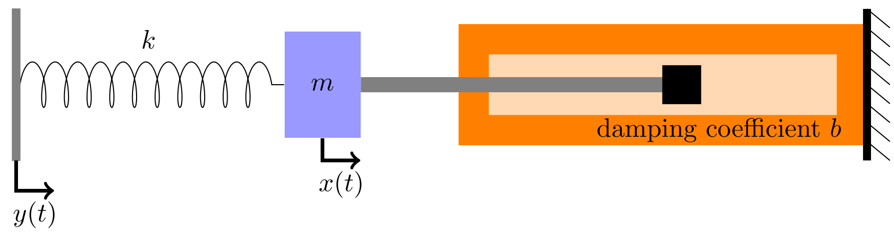
This is always a simple, but tedious, calculation

$$\begin{aligned} P(D)(c_1x_1 + c_2x_2) &= (c_1x_1 + c_2x_2)''' + 6(c_1x_1 + c_2x_2)'' + 7(c_1x_1 + c_2x_2) \\ &= c_1(x_1''' + 6x_1'' + 7x_1) + c_2(x_2''' + 6x_2'' + 7x_2) \\ &= c_1P(D)x_1 + c_2P(D)x_2 \quad \blacksquare \end{aligned}$$

(b) *Say to yourself: "Checking linearity is always easy. You just have to remember to ask."*

Solution: Oh, go on, just say it.

Problem 6.12. *Driving through the spring. Suppose the spring-mass-dashpot is driven by a mechanism that positions the end of the spring at $y(t)$ as shown. As before, $x(t)$ is the position of the mass. We calibrate x and y so that $x = 0$, $y = 0$ is an equilibrium position of the system.*



Give the DE modeling the position $x(t)$ of the mass. Assume, m, k, b, x, y are in compatible units.

Since we control $y(t)$, it is the input. To model $x(t)$, we must consider all the forces on the mass. At time t, the spring is stretched an amount $x(t) - y(t)$, so the spring force is $-k(x - y)$. Likewise, the velocity of the damper through the dashpot is $\dot{x}$. So the damping force is $-b\dot{x}$. Thus, using Newton's second law,

$$m\ddot{x} = -k(x - y) - b\dot{x} \quad \Leftrightarrow \quad m\ddot{x} + b\dot{x} + kx = ky.$$

Problem 6.13. *Let $P(D) = D^2 + 4D + 6I$. Solve $P(D)x = \cos(2t)$.*

Solution: The SRF says a particular solution is given by

$$x_p(t) = \frac{\cos(2t - \phi)}{|P(2i)|}, \text{ where } \phi = \text{Arg}(P(2i)).$$

Computing: $P(2i) = 2 + 8i$, so $|P(2i)| = \sqrt{68}$ and $\boxed{\phi = \text{Arg}(P(2i)) = \tan^{-1}(4) \text{ in Q1}}$.

Thus, $\boxed{x_p(t) = \dfrac{\cos(2t - \phi)}{\sqrt{68}}}$.

The characteristic equation is $r^2 + 4r + 6 = 0$. This has roots $r = -2 \pm \sqrt{2}\,i$.

General homogeneous solution: $\boxed{x_h(t) = c_1e^{-2t}\cos(\sqrt{2}t) + c_2e^{-2t}\sin(\sqrt{2}t)}$.

General solution to the inhomogeneous DE: $\boxed{x(t) = x_p(t) + x_h(t)}$

Problem 7.14. *Find a particular solution to $x'' + x' = t^4$.*
Write down the system of equations for A, B, C, D, E, but don't bother solving.

Solution: We try $x_p = At^5 + Bt^4 + Ct^3 + Dt^2 + Et^1$. (Because the lowest derivative in the DE is x' must increase all degrees in the guess by 1.)

Not showing all the algebra, we have

$$x_p'' + x_p' = 5At^4 + (20A + 4B)t^3 + (12B + 3C)t^2 + (6C + 2D)t + (2D + E) = t^4.$$

Equating coefficients we get the system of equations

$$\begin{array}{rlr} \text{Coeff. of } t^4: & 5A & = 1 \\ \text{Coeff. of } t^3: & 20A{+}4B & = 0 \\ \text{Coeff. of } t^2: & \quad 12B{+}3C & = 0 \\ \text{Coeff. of } t: & \qquad 6C{+}2D & = 0 \\ \text{Coeff. of } 1: & \qquad\quad 2D{+}E & = 0 \end{array}$$

Homogeneous solution: The characteristic equation is $r^2 + r = 0$. This has roots $r = 0, -1$. The general homogeneous solution is $\boxed{x_h(t) = c_1 + c_2e^{-t}.}$

The general solution to the DE is

$$\boxed{x(t) = x_p(t) + x_h(t) = At^5 + Bt^4 + Ct^3 + Dt^2 + Et + c_1 + c_2e^{-t}.}$$

Problem 7.15. *(Example from Topic 7 notes.)*
Solve $y'' + 5y' + 4y = 2t + 3$ by the method of undetermined coefficients.

Solution: First, we find a particular solution using the method of undetermined coefficients:

We guess a trial solution of the form $y_p(t) = At + B$. Our guess has the same degree as the input.

Substitute the guess into the DE and do the algebra to compute the coefficients. Here is one way to present the calculation

$$\begin{array}{rclcl} y_p & = & At & + & B \\ y_p' & = & & & A \\ y_p'' & = & 0 & & \\ y_p'' + 5y_p' + 4y_p & = & 4At & + & (5A + 4B) \end{array}$$

Substituting this into the DE we get:

$$4At + (5A + 4B) = 2t + 3.$$

Now, we equate the coefficients on both sides to get two equations in two unknowns.

$$\begin{array}{lllr} \text{Coefficients of } t: & 4A & & = 2 \\ \text{Coefficients of } 1: & 5A & +4B & = 3 \end{array}$$

This is called a triangular system of equations. First we find $A = 1/2$ and then $B = 1/8$.
So, $\boxed{y_p(t) = \frac{1}{2}t + \frac{1}{8}.}$

Next, we find the solution to the associated homogeneous DE: $y'' + 5y' + 4y = 0$.

Characteristic equation: $r^2 + 5r + 4 = 0 \quad \Rightarrow$ roots are $r = -1, -4$.

General homogeneous solution: $\boxed{y_h(t) = c_1 e^{-t} + c_2 e^{-4t}.}$

Finally, we use the superposition principle to write the general solution to our DE:

$$y(t) = y_p(t) + y_h(t) = \frac{1}{2}t + \frac{1}{8} + c_1 e^{-t} + c_2 e^{-4t}.$$