ES.1803 Problem Section 5, Spring 2024 Solutions

Problem 9.1. *Consider the forced damped system:* $x'' + 2x' + 9x = \cos(\omega t)$*.*

(a) *What is the natural frequency of the system?*

Solution: Natural frequency = frequency of unforced, undamped system = $\sqrt{9} = 3$.

(b) *Find the response of the system in amplitude-phase form.*

Solution: We can go right to the sinusoidal response formula: $\boxed{x_p(t) = \frac{1}{|P(i\omega)|}\cos(\omega t - \phi(\omega))}$,

where, $P(r) = r^2 + 2r + 9 \Rightarrow P(i\omega) = 9 - \omega^2 + 2i\omega$.

So, $\boxed{|P(i\omega)| = \sqrt{(9-\omega^2)^2 + 4\omega^2}}$, and $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}\frac{2\omega}{9-\omega^2} \text{ in Q1 or Q2.}}$

Alternatively (and you should know how to do this)

Complexify: $z'' + 2z' + 9z = e^{i\omega t}$, $x = \text{Re}(z)$.

Char. polynomial: $P(r) = r^2 + 2r + 9$, so $P(i\omega) = 9 - \omega^2 + 2i\omega$.

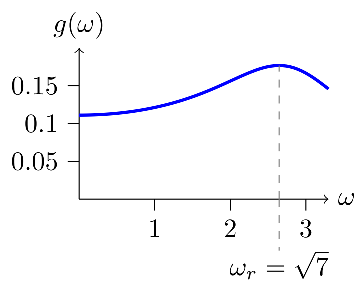
ERF: $z_p(t) = \dfrac{e^{i\omega t}}{P(i\omega)} = \dfrac{e^{i\omega t}}{9 - \omega^2 + 2i\omega}$

Polar form: $9-\omega^2+2i\omega = \sqrt{(9-\omega^2)^2 + 4\omega^2}\,e^{i\phi(\omega)}$, where $\boxed{\phi(\omega) = \tan^{-1}(2\omega/(9-\omega^2)), \text{ in Q1 or Q2.}}$

Thus, $z_p(t) = \dfrac{1}{\sqrt{(9-\omega^2)^2 + 4\omega^2}}e^{i(\omega t - \phi(\omega))}$. This implies $\boxed{x_p(t) = \frac{1}{\sqrt{(9-\omega^2)^2 + 4\omega^2}}\cos(\omega t - \phi(\omega)).}$

(c) *Consider the right hand side of the DE to be the input. What is the amplitude response of the system? Draw its graph –be sure to label your axes correctly*

Solution: Amplitude response = gain = $g(\omega) = \dfrac{1}{\sqrt{(9-\omega^2)^2 + 4\omega^2}}$.



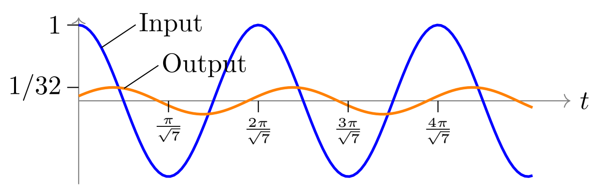
Gain curve and practical resonant frequency

(d) *What is the practical resonant frequency?*

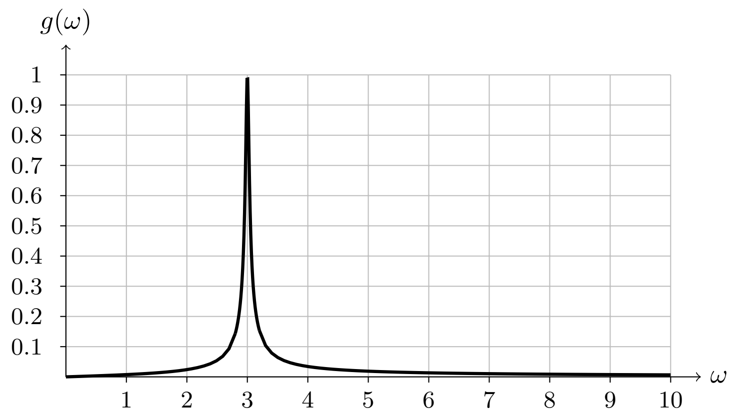
Solution: Practical resonance is where $g(\omega)$ has a maximum. This is the same as where the expression under the radical, $(9-\omega^2)^2 + 4\omega^2$, has a minimum. Simple calculus shows this is at $\omega_r = \sqrt{7}$.

(e) *When* $\omega = \sqrt{7}$ *by how many radians does the output peak lag behind the input peak?*

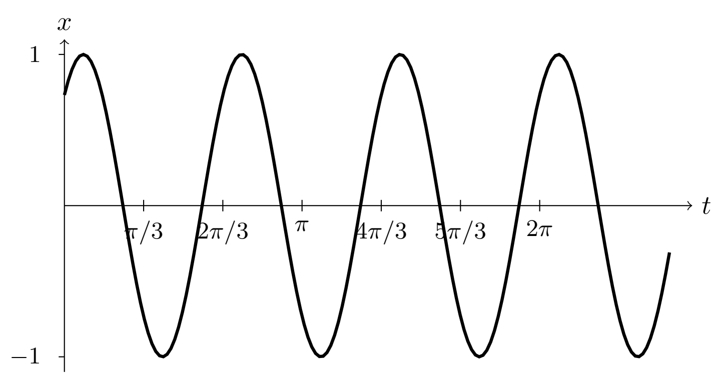
Solution: If $\omega = \sqrt{7}$ then $\phi = \tan^{-1}\sqrt{7} = 1.2$ radians and $g(\sqrt{7}) = 1/\sqrt{32} \approx 0.177$.



Problem 9.2. *Below is a gain curve. Suppose the input is* $\sum_{n=0}^{100} \cos(nt)$. *Give a rough sketch of the output.*



Solution: The gain curve will be 1 near $n = 3$ and near 0 for all other n. Therefore, the system will (essentially) filter out all the frequencies except $n = 3$. The output will look very much like $x = \cos(3t)$.



(In reality, you would see a little wobble due to the small contribution of n near 3.)

Problem 9.3. *Consider the driven first-order system:* $x' + kx = kF_0 \cos(\omega t)$. *We'll take the input to be* $F_0 \cos(\omega t)$. *Solve the DE. Find the amplitude response. Show there is never practical resonance.*

Solution: Use the sinusoidal response formula: $x_p(t) = \dfrac{kF_0 \cos(\omega t - \phi(\omega))}{|P(i\omega)|}$.

Characteristic polynomial: $P(r) = r + k$. So, $P(i\omega) = i\omega + k$.

Polar form: $P(i\omega) = k + i\omega \Rightarrow \boxed{|P(i\omega)| = \sqrt{k^2 + \omega^2}}$, and $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}(\omega/k), \text{ in Q1.}}$

So,

$$\boxed{x_p(t) = \frac{kF_0}{\sqrt{k^2+\omega^2}} \cos(\omega t - \phi(\omega))}$$

Amplitude response = gain = $g(\omega) = \dfrac{k}{\sqrt{k^2+\omega^2}}$. This is a decreasing function in ω, so there is no positive local maximum, i.e., no practical resonance.

Problem 9.4. *Consider the system $x'' + 8x = F_0 \cos(\omega t)$.*

(a) *Why is this called a driven undamped system?*

Solution: This models an undamped harmonic oscillator (with mass = 1). The input $F_0 \cos(\omega t)$ is a force which 'drives' the motion of system.

Note: we assumed that $F_0 \cos(\omega t)$ is the input. Not matter what we call it, the right side of the equation represents a force that drives the system.

(b) *Solve this using the sinusoidal response formula (SRF). Then do it again using complex replacement and the exponential response formula (ERF).*

Solution: Using the SRF and Extended SRF we have

For $\omega \neq \sqrt{8}$: $P(i\omega) = 8 - \omega^2 = |8-\omega^2|e^{i\phi(\omega)}$, where $\phi(\omega) = \begin{cases} 0 & \text{if } 8-\omega^2 > 0 \\ \pi & \text{if } 8-\omega^2 < 0. \end{cases}$

So, $x_p(t) = \dfrac{F_0 \cos(\omega t - \phi(\omega))}{|8-\omega^2|} = \begin{cases} F_0\cos(\omega t) & \text{if } \omega < \sqrt{8} \\ F_0\cos(\omega t - \pi) = -F_0\cos(\omega t) & \text{if } \omega > \sqrt{8}. \end{cases}$

If $\omega = \sqrt{8}$, we need the Extended SRF: $x_p(t) = \dfrac{F_0 t\cos(\omega t - \phi)}{|P'(i\omega)|}$, where $\phi = \text{Arg}(P'(i\omega))$.

$P'(r) = 2r$, so $P'(i\sqrt{8}) = 2i\sqrt{8} = 2\sqrt{8}e^{i\pi/2}$. Therefore,

$$x_p(t) = \frac{F_0 t\cos(\sqrt{8}\,t - \pi/2)}{2\sqrt{8}}.$$

Alternatively, we could use complex replacement to arrive at the same formula.

Complexify: $z'' + 8z = F_0 e^{i\omega t}$, $x = \text{Re}(z)$.

Characteristic polynomial: $P(r) = r^2 + 8$.

Exponential response formula: As above, $P(i\omega) = |P(i\omega)|e^{i\phi}$, where

$$|P(i\omega)| = |8-\omega^2| \quad \text{and } \phi(\omega) = \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } 8-\omega^2 > 0 \\ \pi & \text{if } 8-\omega^2 < 0. \end{cases}$$

So, if $\omega \neq \sqrt{8}$, then

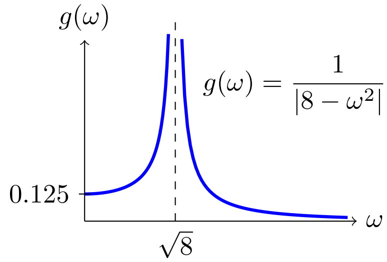
$$z_p(t) = \frac{F_0 e^{i\omega t}}{P(i\omega)} = \frac{F_0 e^{i(\omega t - \phi(\omega))}}{|8-\omega^2|}, \text{ so } x_p(t) = \text{Re}(z(t)) = \frac{F_0\cos(\omega t - \phi(\omega))}{|8-\omega^2|}$$

If $\omega = \sqrt{8}$, then $P(i\omega) = 0$, and we need to use the extended exponential response formula. As usual, we label the natural frequency $\omega_0 = \sqrt{8}$.

$$z_p(t) = \frac{F_0 t e^{i\omega_0 t}}{P'(i\omega_0)} = \frac{F_0 t e^{i\omega_0 t}}{2i\omega_0} = \frac{F_0 t e^{i\omega_0 t}}{2\omega_0 e^{i\pi/2}} = \frac{F_0 t e^{i(\omega_0 t - \pi/2)}}{2\omega_0} \quad \text{so, } x(t) = \text{Re}(z(t)) = \frac{F_0 t\cos(\omega_0 t - \pi/2)}{2\omega_0}.$$

(c) *Consider the right hand side of the DE to be the input. Graph the amplitude response function.*

Solution: $g(\omega) = 1/|8-\omega^2|$. The graph of g vs. ω has a vertical asymptote at $\omega = \sqrt{8}$.



(d) *What is the resonant frequency of the system?*

Solution: Resonant frequency at $\omega = \sqrt{8}$.

(e) *Why is this called the natural frequency?*

Solution: Because the unforced system $x'' + 8x = 0$ will oscillate at this frequency

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Extra problems if time.

Problem 9.5. *Consider the system*

$$2y'' + 10y' + 3y = 3B\cos(\omega t),$$

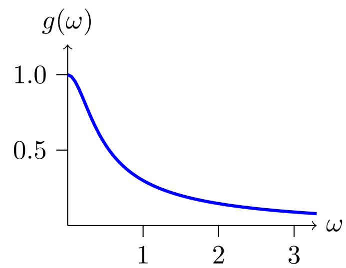
where we consider $B\cos(\omega t)$ to be the input. Find and graph the gain. Find the practical resonant frequency.

Solution: The sinusoidal response formula tells us the sinusoidal solution to this equation is

$$y_p(t) = \frac{3B\cos(\omega t - \phi(\omega))}{|P(i\omega)|}, \qquad \text{where } \phi(\omega) = \operatorname{Arg}(P(i\omega)).$$

Since the input is $B\cos(\omega t)$, the gain is

$$\boxed{g(\omega) = \frac{3}{|P(i\omega)|} = \frac{3}{\sqrt{(3-2\omega^2)^2 + 100\omega^2}}}$$



Finding the practical resonant frequency means finding the value of ω that maximizes the gain:

$$g'(\omega) = -\frac{3}{2} \cdot \frac{-8\omega(3-2\omega^2) + 200\omega}{((3-2\omega^2)^2 + (10\omega)^2)^{3/2}} = 0.$$

Setting the numerator to 0 and solving for ω we find $\omega = 0$ or $\omega = \sqrt{-11}$. Since neither of these is a positive real number we say that there is no practical resonant frequency.

Note: Just to change things up, we computed $g'(\omega)$. We could have used our usual trick and found the minimum of $h(\omega) = \dfrac{1}{g(\omega)^2} = (3-2\omega^2)^2 + 100\omega^2$. Of course, $h'(\omega)$ is just the numerator of $g'(\omega)$.

Problem 9.6. *For the forced undamped system $x'' + 8x = F_0 \cos(\omega t)$, give a detailed description of the phase lag for different input frequencies. (Consider $F_0 \cos(\omega t)$ to be the input.)*

Solution: Because $P(i\omega) = 8 - \omega^2$ is real, we have

$$\phi(\omega) = \begin{cases} 0 & \text{if } 8-\omega^2 < 0, \text{ i.e., } \omega < \omega_0 \\ \pi & \text{if } 8-\omega^2 > 0, \text{ i.e., } \omega > \omega_0. \end{cases}$$

When $\omega = \omega_0$ there is no periodic solution so officially there is no phase lag but, looking at the solution $x_p(t) = t\cos(\omega t - \pi/2)/2\omega_0$, it seems that $\phi = \pi/2$ is a reasonable choice.