ES.1803 Problem Section 7, Spring 2024 Solutions

Problem 13.1. *Compute the following by thinking of matrix multiplication as a linear combination of the columns of the matrix.*

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Solution: Picks out second column $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

Solution: Linear combination: $\quad -2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Problem 13.2. *Is it a vector space? For all of these you just have to check that they are closed under addition and scalar multiplication, i.e. closed under linear combinations.*

(a) *The set of functions $f(x)$ such that $f(5) = 0$.*

Solution: Yes: If $f(5) = 0$ and $g(5) = 0$, then clearly $c_1 f(5) + c_2 g(5) = 0$.

(b) *The set of functions $f(x)$ such that $f(5) = 2$.*

Solution: No: If $f(5) = 2$ and $g(5) = 2$, then $f(5) + g(5) = 4$. So the set is not closed under addition.

(c) *The set of vectors (x, y) in the plane, such that $2x + 3y = 0$.*

Solution: Yes. Graph this, it's a line through the origin. Or, algebraically, if (x_1, y_1) and (x_2, y_2) are in the set, then so is $c_1(x_1, y_1) + c_2(x_2, y_2) = (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$. This is easy to check:

$$2(c_1x_1 + c_2x_2) + 3(c_1y_1 + c_2y_2) = c_1 \underbrace{(2x_1 + 3y_1)}_{\text{equals } 0} + c_2 \underbrace{(2x_2 + 3y_2)}_{\text{equals } 0} = 0$$

(d) *The set of vectors (x, y) in the plane, such that $2x + 3y = 2$.*

Solution: No. Graph this, it's a line NOT through the origin, so it is not closed under scalar multiplication by 0.

Problem 13.3. *Convert the following ODE to a companion system: $x''' + 2x'' + 3x' + 4x = \cos(5t)$.*

Solution: Let $y = x'$, $z = x''$. So, $z' + 2z + 3y + 4x = \cos(5t)$. We get $z' = -4x - 3y - 2z + \cos(5t)$. So,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(5t) \end{bmatrix}$$

Problem 14.4. *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 3 & 6 & 10 & 3 & 6 \end{bmatrix}$. *Put A in row reduced echelon form. Find the rank, a basis of the column space, a basis of the null space, and the dimension of each of the spaces.*

Solution: Here are the row reduction steps:

$$A \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The pivot columns are Columns 1 and 3. These give a basis for the column space of A.

$$\text{Basis: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\} \quad \text{Col}(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\}$$

Note the semantic distinction: the basis set contains 2 vectors, the column space is a set with infinitely many vectors.

Rank(A) = # of pivots = dimension of column space = 2.

The null space of A has dimension 3 = # of free variables. Since A and R have the same null space, we work with R.

We find a basis two ways. First, we solve $R\mathbf{x} = \mathbf{0}$ by writing matrix multiplication as a linear combination of columns.

$$R\mathbf{x} = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_4 + 2x_5 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving for the pivot variables, these equations show, $x_1 = -2x_2 - x_4 - 2x_5$ and $x_3 = 0$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 - 2x_5 \\ x_2 \\ 0 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives us 3 basis vectors:

$$\mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So,

$$\text{Null}(A) = \text{ span of } \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A faster method, is to set, in turn, each free variable to 1 and the others to 0 and then solve for the pivot variables. We do the computation by putting the values below the RREF matrix R.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{array}$$

This gives the same basis we found with our first method.

Problem 14.5. *Let* $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Suppose R is the row reduced echelon form for A.*

(a) *What is the rank of A?*

Solution: A and R have the same rank. Two pivots in R implies rank $= 2$.

(b) *Find a basis for the null space of A.*

Solution: A and R have the same null space. The second and fourth variables are free. We find a basis by setting them to 1 and 0 in turn and then solving for the pivot variables.

Note, I find this computation easiest to do if I think of matrix multiplication as a combination of the columns

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

We show the computation by putting the variables below the matrix. Each row below the matrix shows one solution to $R\mathbf{x} = \mathbf{0}$, found by setting one free variable to 1, the other free variables to 0 and solving for the pivot variables.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -1 & 1 \end{array}$$

So a basis of Null(A) contains the two vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

There are, of course, many other bases. Our standard algorithm produces the one given.

(c) *Suppose the column space of A has basis* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. *Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.*

Solution: Looking at R the Columns 1 and 3 are pivot columns. We put the given basis in those columns:

$$A = \begin{bmatrix} 1 & * & 3 & * \\ 1 & * & 1 & * \\ 0 & * & 1 & * \end{bmatrix}$$

The free columns of R are linear combinations of the pivot columns and those of A are the same linear combinations. In R it is clear that

$$\text{Col}_2 = 2 \times \text{Col}_1 \text{ and } \text{Col}_4 = 3 \times \text{Col}_1 + \text{Col}_3.$$

So,

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(d) *Find a matrix with the same row reduced echelon form, but such that* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *are in its column space.*

Solution: We found the relationships between the columns in Part (c). So we put the given columns as pivot columns and construct the free columns from these relationships: $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$

Note: you could put any other basis for the subspace generated by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the pivot columns and adjust the free columns accordingly.

Extra problems if time.

Problem 14.6. *Suppose we want to solve $A\mathbf{x} = \mathbf{b}$, where* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

(a) *When is this possible? Answer this in the form: "**b** must be a linear combination of the two vectors ..."*

Solution: The equation can be solved exactly when **b** is a linear combination of the columns of A, i.e., when **b** is in Col(A). We can find a basis of Col(A) by finding the RREF. The RREF is

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first two columns are pivot columns and the last one is free. Thus, the column space of A is spanned by the first two columns of A. Since, **b** must be in Col(A), we can answer the question as follows:

b must be a linear combination of the first two columns of A, i.e., $\mathbf{b} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Note. Since there are lots of other possible bases for the column space, this is just one of many possible answers.

(b) *$A\mathbf{x} = \mathbf{b}$ is certainly solvable for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.*

Solution: $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the first column of A, so the obvious solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We can take this (or any other solution!) as $\mathbf{x_p}$. To get the general solution, we must add the general homogeneous solution.

The homogeneous solution is the same as the null space of A or R. We can find that by setting the free variable $x_3 = 1$ and solving for $x_1 = 1$ and $x_2 = -2$. So, $\mathbf{x_h} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + c \\ -2c \\ c \end{bmatrix}.$$

Problem 14.7. *Suppose that the row reduced echelon form of the 4×6 matrix B is*

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) *Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.*

Solution: This is just another way of asking for a basis of Null(B). The null space of B is the same as the null space of its row-echelon form.

The free variables are: x_1, x_3 and x_6. As usual, we set them to 1 in turn.

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & -7 & -9 & 1 \end{matrix}$$

Thus a basis of Null(B) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ -7 \\ -9 \\ 1 \end{bmatrix} \right\}$$

(b) *Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.*

Solution: $\mathbf{b_1}$ must be $\mathbf{0}$, since applying row operations to it give $\mathbf{0}$, and row operations are reversible.

$\mathbf{b_2} \neq \mathbf{0}$. This is all we can say.

The linear relations among the columns of B are the linear relations among the columns of R: so the columns of B corresponding to the pivot columns of R are independent of the previous columns: $\mathbf{b_2}$, $\mathbf{b_4}$, and $\mathbf{b_5}$.

For the linear relations, we just copy what we know for R: $\mathbf{b_1} = 0$; $\mathbf{b_3} = 3\mathbf{b_2}$; $\mathbf{b_6} = 5\mathbf{b_2} + 7\mathbf{b_4} + 9\mathbf{b_5}$.

By a 'linearly independent set of vectors of which every vector in the column space is a linear combination' we just mean a basis of Col(B), These are the pivot columns. That is, $\{\mathbf{b_2}, \mathbf{b_4}, \mathbf{b_5}\}$ is a basis for the column space of B.

Problem 14.8. *Suppose we have a matrix equation*

$$\begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Solution: To have a nontrivial null space the rank must be less than 3. Since the first and third columns are independent, the middle column must be a linear combination of them. Geometrically, the middle column is in the plane containing the origin and the other two columns.

Problem 14.9. *For what values of y is it the case that the columns of $\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix}$ form a linearly independent set?*

Solution: The columns are linearly independent when the matrix has rank 3. We can find the rank by row reduction:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & y-3 & -2 \\ 0 & -4 & -4 \end{bmatrix} \xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -4 \\ 0 & y-3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 = -R_2/4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & y-3 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 - (y-3)R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-y \end{bmatrix}$$

If $1 - y \neq 0$, then we have 3 pivots. So the columns are linearly independent exactly when $y \neq 1$.

Problem 14.10. *For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$:*

(a) *Find the row reduced echelon form of A; call it R.*

Solution: Here are the row reduction steps:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow[R_3 = R_3 + 3R_2]{R_2 = -R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

(b) *The last column of R should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of R. Find a vector $\mathbf{x}$, such that $R\mathbf{x} = \mathbf{0}$, which expresses this linear relationship.*

Solution: This is just an awkward way of asking about null vectors. The first two columns are pivotal and the third is free. So the third is a combination of the other two. By inspection of R, we see that

$$\text{Col}_3 = -\text{Col}_1 + 2\text{Col}_2 \quad \Rightarrow \text{Col}_1 - 2\text{Col}_2 + \text{Col}_3 = \mathbf{0}.$$

As a matrix equation this is $R \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) *Verify that the same relationship holds among the columns of A.*

Solution: The third column is indeed minus the first plus twice the second. As a matrix equation,

$$A\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(d) *Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.*

Solution: Row transformations do the same thing to the entries of all columns.

This is the same as saying that if A and B are related by row-operations, then their null spaces coincide. Their column spaces usually do not.

Problem 14.11. *Consider the following system of equations:*

$$\begin{aligned} x + \ \ y + \ \ z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) *Write this system of equations as a matrix equation.*

Solution:

$$\begin{bmatrix}1&1&1\\1&2&3\\1&3&6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\7\\11\end{bmatrix}$$

(b) *Use row reduction to get to row echelon form. What is the solution set?*

Solution: Set up the augmented matrix:

$$\left[\begin{array}{ccc|c}1&1&1&5\\1&2&3&7\\1&3&6&11\end{array}\right]$$

Do row reduction to RREF

$$\left[\begin{array}{ccc|c}1&1&1&5\\1&2&3&7\\1&3&6&11\end{array}\right] \xrightarrow[\text{Row}_3 = \text{Row}_3 - \text{Row}_1]{\text{Row}_2 = \text{Row}_2 - \text{Row}_1} \left[\begin{array}{ccc|c}1&1&1&5\\0&1&2&2\\0&2&5&6\end{array}\right] \xrightarrow{\text{Row}_3 = \text{Row}_3 - 2\text{Row}_2} \left[\begin{array}{ccc|c}1&1&1&5\\0&1&2&2\\0&0&1&2\end{array}\right]$$

$$\xrightarrow[\text{Row}_1 = \text{Row}_1 - \text{Row}_3]{\text{Row}_2 = \text{Row}_2 - 2\text{Row}_3} \left[\begin{array}{ccc|c}1&1&0&3\\0&1&0&-2\\0&0&1&2\end{array}\right] \xrightarrow{\text{Row}_1 = \text{Row}_1 - \text{Row}_2} \left[\begin{array}{ccc|c}1&0&0&5\\0&1&0&-2\\0&0&1&2\end{array}\right]$$

The solution is $x = 5$, $y = -2$, $z = 2$. You can check this by substituting it into the original equations.

Problem 14.12. **(a)** *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & x \end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

Solution: Each column of the product is a multiple of the column vector in the first factor. The 1s show that they are the same multiple. So x must be 2.

Alternatively, each row of the product is a multiple of the row vector. The first column of the product shows that the second row must be twice the first row. So x must be 2.

One equation that works for $x = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

None of the $\bullet$'s can be 0, since that would make the corresponding row or column in the product $\mathbf{0}$.

(b) *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet & 3 \\ \bullet & 4 \\ \bullet & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the $\bullet$'s?

Solution: The matrix equation says: the first column plus twice the second column is zero. So the first column must be $\begin{bmatrix} -6 \\ -8 \\ -10 \end{bmatrix}$.

(c) *Suppose we have a matrix equation*

$$\begin{bmatrix} x & 3 \\ y & 4 \\ z & 5 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

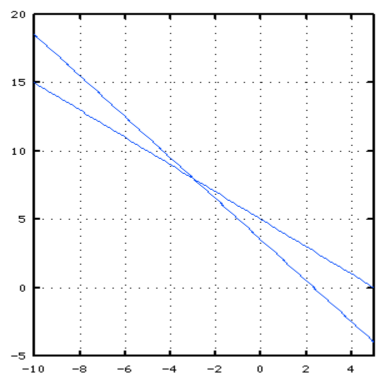
Solution: The equation says that the columns of the matrix form a linearly dependent set. That is: one is a multiple of the other. Since the second column is nonzero, we can be sure that the first is a multiple of the second: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ for some t.

Problem 14.13. *Solve this system of linear equations. How many methods can you think of to solve this system?*

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Solution: Some ideas:
(1) Graphically with interesecting lines.
(2) Elimination.
(3) Row reduce the augmented matrix.
(4) Matrix inverse.

(1) $y = -x + 5$ and $y = \frac{7}{2} - \frac{3}{2}x$ are two straight lines of different slopes; so they meet at a single point. To find where, we could eyeball the picture—maybe $(-3, 8)$? That satisfies both equations!



(2) We can use elimination: Subtract 3 times the first equation from the second. Retaining the first equation as well, we get

$$\begin{aligned} x + y &= 5 \\ 0 - y &= -8 \end{aligned}$$

and then the first equation gives $x = -3$. In fact, as a second step, we could add the new second equation to the first one:

$$\begin{aligned} x + 0 &= -3 \\ 0 - y &= -8 \end{aligned}$$

Thus $(x, y) = (-3, 8)$ is the solution.

(3) Matrix methods: The system is $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

Again $(x, y) = (-3, 8)$ is the solution.

Problem 14.14. *Solve the following equation using row reduction:*

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) *At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?*

Solution: The augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right]$.

Do row reduction:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{\text{Row}_2 = \text{Row}_2 - 3\text{Row}_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -3 \end{array}\right] \xrightarrow{\text{Row}_2 = -\text{Row}_2/3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The last equation now reads $0x + 0y = 1$, which is rather hard to satisfy.

(We could already see this problem after the first reduction step.)

The last column was pivotal. This implies there is a row in the augmented RREF matrix with all zeros except for a 1 in the last column. This row corresponds to the equation $0x + 0y = 1$, which explains why there are no solutions.

(b) *Find a new vector* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *such that* $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *has a solution.*

Solution: Well, we could always take $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because the equation is then obviously solved by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To be more general, we can take $\mathbf{b}$ in the column space of the coefficient matrix. The row reduced echelon form shows that Column 1 is the only pivot column. So the column space has basis $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Thus, the vectors $c \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are exactly the vectors for which the equation admits a solution.

Problem 14.15. *Show that the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *corresponds to counter-clockwise rotation about the origin by 90 degrees, by computing the effect of this matrix on the vectors* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ *and drawing* $\mathbf{v}_1$, $\mathbf{v}_2$, $A\mathbf{v}_1$, $A\mathbf{v}_2$ *on the plane.*

Solution: It's easy to compute:

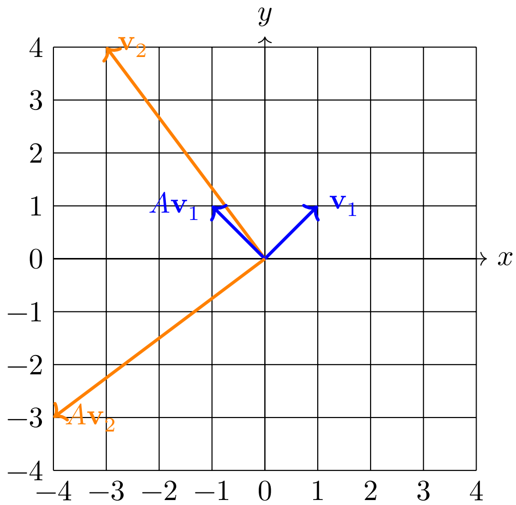
$$A\mathbf{v}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } A\mathbf{v}_2 = \begin{bmatrix} -4 \\ -3 \end{bmatrix}.$$

Using the dot product we can check that $\mathbf{v}_1$ is orthogonal to $A\mathbf{v}_1$:

$$\mathbf{v}_1 \cdot A\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.$$

Similarly $\mathbf{v}_2 \cdot A\mathbf{v_2} = 0$.

This shows hat tA has rotated each of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by 90°. The figure shows the rotation is counter-clockwise.



Problem 13.16. *Make up a block matrix problem: Multiply a 4×4 matrix made up of four 2×2 blocks (two blocks of 0s, one block = identity, one block something else) times a 4×2 matrix with (i.e., two 2×2 blocks)*

Solution: You do this!