# ES.1803 Problem Section 8, Spring 2024 Solutions

**Problem 17.1.** **(a)** *Let* $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. *Solve* $\mathbf{x}' = A\mathbf{x}$.

**Solution:** Characteristic equation: $|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 10 = 0$. So the eigenvalues are $\lambda = 2, -5$.

Basic Eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda = 2$: $\quad A - \lambda I = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$\lambda = -5$: $\quad A - \lambda I = \begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Two (modal) solutions: $\quad \mathbf{x_1}(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\quad \mathbf{x_2}(t) = e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

General solution: $\mathbf{x}(t) = c_1 \mathbf{x_1}(t) + c_2 \mathbf{x_2}(t)$.

**(b)** *What is the solution to* $\mathbf{x}' = A\mathbf{x}$ *with* $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

**Solution:** We use the initial condition to find values for the parameters $c_1$, $c_2$

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

In matrix form we have $\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6/7 \\ -4/7 \end{bmatrix}.$$

Thus, $\mathbf{x}(t) = \frac{6}{7} e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{4}{7} e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

**(c)** *Decouple the system in Part (a). That is, make a change of variables that converts the system to a decoupled one. Write the system in the new variables.*

**Solution:** The decoupling change of variables is $\begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} u \\ v \end{bmatrix}$, where $S$ is the matrix of eigenvectors. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} x &= 3u + v \\ y &= 2u + 3v \end{cases}$$

In these variables the system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \Lambda \begin{bmatrix} u \\ v \end{bmatrix}$, where $\Lambda$ is the diagonal matrix of eigenvalues. That is,

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} u' &= 2u \\ v' &= -5v \end{cases}$$

**Problem 16.2.** **(a)** *Find the eigenvalues and basic eigenvectors of* $A = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

**Solution:** Characteristic equation: $\begin{vmatrix} 3-\lambda & 1 & -3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-2)(\lambda-3) = 0$: eigenvalues are $3, 3, 2$. (Since this is a triangular matrix, you should be able to get these values by inspection.)

The eigenspace corresponding to an eigenvalue $\lambda$ is $\text{Null}(A - \lambda I)$. We'll need row reduction to find this for each $\lambda$.

$\lambda = 3$: $\quad (A - 3I) = \begin{bmatrix} 0 & 1 & -3 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row}_2 = \text{Row}_2 + \text{Row}_1} \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

There are two free variables (first and third). We use our usual algorithm and notation to find a basis for the null space:

$$\begin{array}{ccc} \left[\begin{matrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right] \\ \begin{matrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{matrix} \end{array}$$

Our two basis vectors are: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. These are two independent eigenvectors with eigenvalue 3.

For $\lambda = 2$, we won't show the row reduction steps.

$$(A - 2I) = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second column is free. Again, we make our usual computation to find a basis.

$$\begin{array}{c} \left[\begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}\right] \\ \begin{matrix} x_1 & x_2 & x_3 \\ -1 & 1 & 0 \end{matrix} \end{array}$$

Our basic eigenvector is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

**(b)** *Write A in diagonalized form.*

**Solution:** Let $\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (diagonal matrix of eigenvalues).

Let $S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (matrix of eigenvectors)

Note: The eigenvectors in $S$ must be in the same order as the eigenvalues in $\Lambda$.

We know $A = S\Lambda S^{-1}$. This is the diagonalized form for $A$.

**(c)** *Compute $A^5$.*

**Solution:** $A^5 = S\Lambda^5 S^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 243 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$

**Problem 15.3.** **(a)** *Use row reduction to find the inverse of the matrix $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.*

**(b)** *Use the record of the row operations to compute the determinant of $A$*

**(a) Solution:** Augment the $A$ by the identity and then use row operations to reduce the $A$ to the identity.

$$\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1 \text{ and } R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right]$$
$$\xrightarrow{R_2 - 6R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -7 & 1 & -6 \end{array}\right]$$
$$\xrightarrow{\text{scale } R_2 \text{ by -1/7}} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$
$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2/7 & -5/7 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 2/7 & -5/7 \\ -1/7 & 6/7 \end{bmatrix}$

**(b)** The only operations that change the determinant are swapping and scaling. In this case, there is one swap and one scale by $-1/7$. The row reduction starts with $A$ and ends with $I$, so

$$1 = \det(I) = (-1/7) \cdot (-1) \cdot \det(A) \Rightarrow \boxed{\det(A) = 7}.$$

**Problem 16.4.** *Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.*

**(a)** *What are the eigenvalues of $A$?*

**Solution:** For an upper triangular matrix the eigenvalues are the diagonal entries: $a$, 2, 3.

**(b)** *For what value (or values) of $a, b, c, e$ is $A$ singular (non-invertible)?*

**Solution:** $\det(A)$ = product of eigenvalues. So $A$ is singular when $a = 0$. The parameters $b, c, e$ can take any values.

**(c)** *What is the minimum rank of $A$ (as $a, b, c, e$ vary)? What's the maximum?*

**Solution:** When $a = 0$, the null space is dimension 1, so rank =2.

When $a \neq 0$, $A$ is invertible, so has rank = 3.

**(d)** *Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.*

**Solution:** The two positive eigenvalues imply the system is unstable.

**Problem 16.5.** *Suppose that* $A = S\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}S^{-1}$.

**(a)** *What are the eigenvalues of $A$?*

**Solution:** The eigenvalues are the same as the diagonal matrix, i.e., 1, 2, 3.

**(b)** *Express $A^2$ and $A^{-1}$ in terms of $S$.*

**Solution:** $A^2 = S\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 9\end{bmatrix}S^{-1}$; $\quad A^{-1} = S\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/3\end{bmatrix}S^{-1}$.

**(c)** *What would I need to know about $S$ in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?*

**Solution:** You need to know the eigenvector that goes with the eigenvalue 3. That is, you need to know the third column of $S$.

**Problem 17.6.** ***(Complex roots)*** *Solve* $\mathbf{x}' = \begin{bmatrix}7 & -5\\ 4 & 3\end{bmatrix}\mathbf{x}$ *for the general real-valued solution.*

**Solution:** Coefficient matrix: $\quad A = \begin{bmatrix}7 & -5\\ 4 & 3\end{bmatrix}$.

Characteristic equation: $\quad \det(A - \lambda I) = \begin{vmatrix}7-\lambda & -5\\ 4 & 3-\lambda\end{vmatrix} = \lambda^2 - 10\lambda + 41 = 0.$

Eigenvalues: $\quad \lambda = 5 \pm 4i$.

Basic eigenvectors (basis of $\text{Null}(A - \lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 5 + 4i$: $\quad A - \lambda I = \begin{bmatrix}2-4i & -5\\ 4 & -2-4i\end{bmatrix}\mathbf{v} = \begin{bmatrix}0\\ 0\end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix}5\\ 2-4i\end{bmatrix}$.

Complex solution:

$$\mathbf{z}(t) = e^{(5+4i)t}\begin{bmatrix}5\\ 2-4i\end{bmatrix} = e^{5t}\begin{bmatrix}5\cos(4t) + i5\sin(4t)\\ 2\cos(4t) + 4\sin(4t) + i(-4\cos(4t) + 2\sin(4t))\end{bmatrix}.$$

Both real and imaginary parts are solutions to the DE:

$$\mathbf{x_1}(t) = e^{5t}\begin{bmatrix}5\cos(4t)\\ 2\cos(4t) + 4\sin(4t)\end{bmatrix}, \quad \mathbf{x_2}(t) = e^{5t}\begin{bmatrix}5\sin(4t)\\ -4\cos(4t) + 2\sin(4t)\end{bmatrix}$$

General real-valued solution (by superposition): $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

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**Extra problems if time.**

**Problem 17.7.** *Solve $x' = -3x + y, \quad y' = 2x - 2y$.*

**Solution:** The coefficient matrix is $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 5\lambda + 4 = 0$. This has roots $\lambda = -1, -4$.

Basic eigenvectors (basis of Null$(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = -1$: $A - \lambda I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. Basic eigenvector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\lambda = -4$: $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Basic eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Modal solutions: $\mathbf{x_1}(t) = e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x_2}(t) = e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

General solution $\mathbf{x}(\mathbf{t}) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) = c_1e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Problem 16.8.** **(b)** *Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.*

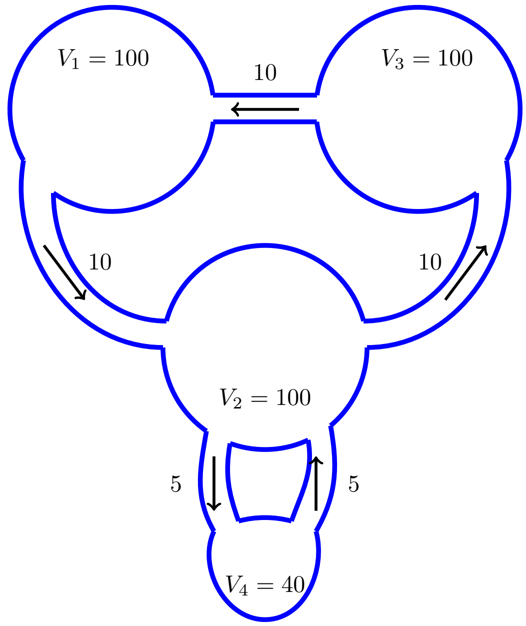
**Solution:** Characteristic equation: $\begin{vmatrix} -3-\lambda & 4 \\ 2 & -5-\lambda \end{vmatrix} = \lambda^2 + 8\lambda + 7 = 0 \Rightarrow \lambda = -1, -7$.

Basic eigenvectors for the eigenvalue $\lambda$ are a basis of Null$(A - \lambda I)$. That is, basic solutions to $(A - \lambda I)\mathbf{v} = 0$. For the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda_1 = -1$: $(A - \lambda I) = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$. Basic eigenvector: $\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda_2 = -7$: $(A - 7I) = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$. Basic eigenvector: $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Problem 17.9.** *The following figure shows a closed tank system with volumes and flows as indicated (in compatible units). Let's call the tank with $V_1 = 100$ tank 1, etc.*




**(a)** *Write down a system of differential equations modeling the amount of solute in each tank.*

**Solution:** Let $x_1$, $x_2$, $x_3$, $x_4$ be the amount of solute in tanks 1 to 4 respectively. Note that the system is balanced, in that the volume in each tank is not changing. Using rate = rate in - rate out we get the following equations.

$$
\begin{aligned}
x_1' &= -10\frac{x_1}{V_1} + 10\frac{x_3}{V_3} = -0.1x_1 + 0.1x_3 \\
x_2' &= 10\frac{x_1}{V_1} - 15\frac{x_2}{V_2} + 5\frac{x_4}{V_4} = 0.1x_1 + -0.15x_2 + 0.125x_4 \\
x_3' &= 10\frac{x_2}{V_2} - 10\frac{x_3}{V_3} = 0.1x_2 - 0.1x_3 \\
x_4' &= 5\frac{x_2}{V_2} - 5\frac{x_4}{V_4} = 0.05x_2 - 0.125x_4
\end{aligned}
$$

In matrix form this is

$$
\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} -0.1 & 0 & 0.1 & 0 \\ 0.1 & -0.15 & 0 & 0.125 \\ 0 & 0.1 & -0.1 & 0 \\ 0 & 0.05 & 0 & -0.125 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

**(b)** *Without computation you know one eigenvalue. What is it? What is a corresponding eigenvector?*

**Solution:** Eventually the system has to reach equilibrium, where all the concentrations are equal. This means one eigenvalue is 0. At equilibrium we must have

$$\frac{x_1}{V_1} = \frac{x_2}{V_2} = \frac{x_3}{V_3} = \frac{x_4}{V_4}.$$

Therefore, using the values for the volumes, we have

$$x_2 = x_1, \quad x_3 = x_1, \quad x_4 = 0.4x_1.$$

We have $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0.4 \end{bmatrix}$ is an eigenvector.

**(c)** *What can you say about all the other eigenvalues?*

**Solution:** They all must be negative, or complex with negative real part. If any were positive, the amount of solute would be growing, which is impossible in a closed system.

**Problem 16.10.** *Suppose that the matrix $B$ has eigenvalues 1 and 7, with eigenvectors*

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

*respectively.*

**(a)** *What is the solution to $\mathbf{x}' = Bx$ with $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?*

**Solution:** The general solution is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

We use the initial condition to find $c_1$ and $c_2$:

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

In matrix form this is $\begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

So, $\mathbf{x}(t) = \frac{1}{3} e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3} e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

**(b)** *Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.*

**Solution:** Decoupling is just the change of variables $\mathbf{u} = S^{-1}\mathbf{x}$. So,

$$\mathbf{u} = S^{-1}\mathbf{x} \;\Leftrightarrow\; \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \;\Leftrightarrow\; \begin{cases} u = x/6 - 5x/6 \\ v = x/6 + y/6. \end{cases}$$

In these coordinates the decoupled system is $\mathbf{u}' = \Lambda \mathbf{u} \;\Leftrightarrow\; \begin{bmatrix} u' \\ v' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

**(c)** *Give an argument based on transformations why $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given above.*

Using the definition of eigenvalues and eigenvectors, we need to show that

$$B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Multiplying by a standard basis vector just picks out the corresponding column of a matrix. So we have the following multiplication table:

$$\begin{aligned} S\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}1\\-1\end{bmatrix} \quad \Rightarrow S^{-1}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \\ S\begin{bmatrix}0\\1\end{bmatrix} &= \begin{bmatrix}5\\1\end{bmatrix} \quad \Rightarrow S^{-1}\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \\ \Lambda\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}1\\0\end{bmatrix} \\ \Lambda\begin{bmatrix}0\\1\end{bmatrix} &= 7\begin{bmatrix}0\\1\end{bmatrix}. \end{aligned}$$

Using this table, we can now compute the product $S\Lambda S^{-1}\begin{bmatrix}5\\1\end{bmatrix}$.

$$S\Lambda S^{-1}\begin{bmatrix}5\\1\end{bmatrix} = S\Lambda\begin{bmatrix}0\\1\end{bmatrix} = S\begin{bmatrix}0\\7\end{bmatrix} = 7S\begin{bmatrix}0\\1\end{bmatrix} = 7\begin{bmatrix}5\\1\end{bmatrix}.$$

This shows that $\begin{bmatrix}5\\1\end{bmatrix}$ is an eigenvector of $S\Lambda S^{-1}$ with eigenvalue 7. The other eigenvalue/eigenvector pair behaves the same way.

**Problem 16.11.** *Suppose the $2 \times 2$ matrix $A$ has eigenvectors $\mathbf{v_1} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix}1\\3\end{bmatrix}$ with eigenvalues 2 and 4 respectively.*

**(a)** *Find $A(\mathbf{v_1} + \mathbf{v_2})$.*

**Solution:** $A(\mathbf{v_1} + \mathbf{v_2}) = A\mathbf{v_1} + A\mathbf{v_2} = 2\mathbf{v_1} + 4\mathbf{v_2} = 2\begin{bmatrix}1\\2\end{bmatrix} + 4\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}6\\16\end{bmatrix}$.

**(b)** *Find $A(\mathbf{5v_1} + \mathbf{6v_2})$.*

**Solution:** $A(\mathbf{5v_1} + \mathbf{6v_2}) = 5A\mathbf{v_1} + 6A\mathbf{v_2} = 10\mathbf{v_1} + 24\mathbf{v_2} = 10\begin{bmatrix}1\\2\end{bmatrix} + 24\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}34\\92\end{bmatrix}$.

**(c)** *Find $A\begin{bmatrix}4\\9\end{bmatrix}$*

**Solution:** By inspection or solving some equations, we get $\begin{bmatrix}4\\9\end{bmatrix} = 3\mathbf{v_1} + \mathbf{v_2}$. So,

$$A\begin{bmatrix}4\\9\end{bmatrix} = 3A\mathbf{v_1} + A\mathbf{v_2} = 6\begin{bmatrix}1\\2\end{bmatrix} + 4\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}10\\24\end{bmatrix}.$$

**Problem 16.12.** **(a)** *Without calculation, find the eigenvalues and and basic eigenvectors for $A = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$.*

**Solution:** Since $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$, we have $\begin{bmatrix}1\\0\end{bmatrix}$ is an eigenvector with eigenvalue 2.

Likewise, $\begin{bmatrix}0\\1\end{bmatrix}$ is an eigenvector with eigenvalue 3.

**(b)** *Without calculation, find at least one eigenvector and eigenvalue for* $A = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$.

**Solution:** Since $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$, we have $\begin{bmatrix}1\\0\end{bmatrix}$ is an eigenvector with eigenvalue 2.

The second eigenvector requires a small calculation.

**Problem 16.13.** *Find the eigenvalues and basic eigenvectors of* $A = \begin{bmatrix}-3 & 13\\-2 & -1\end{bmatrix}$.

**Solution:** Characteristic equation: $\begin{vmatrix}-3-\lambda & 13\\-2 & -1-\lambda\end{vmatrix} = \lambda^2 + 4\lambda + 29 = 0 \Rightarrow \lambda = -2 \pm 5i$.

Basic eigenvectors for $\lambda$ are a basis of $\text{Null}(A - \lambda I)$.

$\lambda_1 = -2+5i$: $(A - \lambda_1 I)\mathbf{v} = \begin{bmatrix}-1-5i & 13\\-2 & 1-5i\end{bmatrix}\mathbf{v_1} = 0$. Basic eigenvector: $\mathbf{v_1} = \begin{bmatrix}13\\1+5i\end{bmatrix}$.

$\lambda_2 = -2-5i$: Use complex conjugate: $\mathbf{v_2} = \overline{\mathbf{v}}_\mathbf{1} = \begin{bmatrix}13\\1-5i\end{bmatrix}$.

**Problem 17.14.** *Solve the system* $x' = x + 2y;\quad y' = -2x + y$.

**Solution:** The coefficient matrix is $A = \begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix}1-\lambda & 2\\-2 & 1-\lambda\end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0$.

Eigenvalues $1 \pm 2i$.

Basic eigenvectors (basis of $\text{Null}(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 1+2i$: $A - \lambda I = \begin{bmatrix}-2i & 2\\-2 & -2i\end{bmatrix}$. Basic eigenvector $\begin{bmatrix}1\\i\end{bmatrix}$.

(We don't need an eigenvector from the complex conjugate $\lambda = 1 - 2i$.)

Complex solution: $\mathbf{z}(t) = e^{(1+2i)t}\begin{bmatrix}1\\i\end{bmatrix} = e^t(\cos(2t)+i\sin(2t))\begin{bmatrix}1\\i\end{bmatrix} = e^t\begin{bmatrix}\cos(2t)+i\sin(2t)\\-\sin(2t)+i\cos(2t)\end{bmatrix}$.

The real and imaginary parts of $\mathbf{z}$ are both solutions:

$\mathbf{x_1}(t) = e^t\begin{bmatrix}\cos(2t)\\-\sin(2t)\end{bmatrix}$, $\mathbf{x_2}(t) = e^t\begin{bmatrix}\sin(2t)\\\cos(2t)\end{bmatrix}$.

General solution: $\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 e^t\begin{bmatrix}\cos(2t)\\-\sin(2t)\end{bmatrix} + c_2 e^t\begin{bmatrix}\sin(2t)\\\cos(2t)\end{bmatrix}$.

Or $x(t) = c_1 e^t\cos(2t) + c_2 e^t\sin(2t);\quad y(t) = -c_1 e^t\sin(2t) + c_2 e^t\cos(2t)$.

**Problem 17.15.** ***(Repeated roots)*** *Solve* $\mathbf{x}' = \begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix}\mathbf{x}$.

**Solution:** The coefficient matrix is $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$

Eigenvalues: $\lambda = 2, 2$ (repeated)

Basic eigenvectors (basis of Null$(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 2$: $A - \lambda I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Basic eigenvector $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

This gives one modal solution: $\mathbf{x_1}(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Since there are not enough independent eigenvectors, the system is defective. For the second solution, we look for one of the form

$$\mathbf{x_2} = te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\mathbf{w}.$$

($\mathbf{w}$ is called a generalized eigenvector. It satisfies $(A - 2I)\mathbf{w} = \mathbf{v}$.)

After some algebra, we find that we can take $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, $\mathbf{x_2}(t) = te^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

General solution: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

**Problem 15.16.** *Use row reduction to find inverses of the following matrices. As you do this, record the row operations carefully for later problems.*

**(a)** $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & 2 & -2 \end{bmatrix}$

**Solution:**

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -6 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 + 6R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & -2 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 10 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3/(-2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 1 & -1/2 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 1 & -1/2 \end{bmatrix}$

**(b)** $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & 2 \\ 3 & 5 & 7 \end{bmatrix}$

**Solution:**

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -4 & -8 & -2 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -4 & -8 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

The last row of the reduced matrix has all zeros on the left. This implies the rank of $B$ is 2, and therefore no inverse exists.

**(c)** $C = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

**Solution:** No inverse the matrix is not square.

**(d)** $D = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 3 & 2 & 5 \end{bmatrix}$

**Solution:** Row reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 5 & -1 & 1 & 0 \\ 0 & -4 & -4 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 5 & -1 & 1 & 0 \\ 0 & 0 & -14 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3/(-14)]{R_2 = R_2/(-2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/14 & 1/7 & -1/14 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + \frac{5}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 11/14 & -3/7 & 3/14 \\ 0 & 1 & 0 & 19/28 & -1/7 & -5/28 \\ 0 & 0 & 1 & 1/14 & 1/7 & -1/14 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/7 & -1/7 & 4/7 \\ 0 & 1 & 0 & 19/28 & -1/7 & -5/28 \\ 0 & 0 & 1 & 1/14 & 1/7 & -1/14 \end{array}\right]$$

So $D^{-1} = \begin{bmatrix} -4/7 & -1/7 & 4/7 \\ 19/28 & -1/7 & -5/28 \\ 1/14 & 1/7 & -1/14 \end{bmatrix} = \dfrac{1}{28} \begin{bmatrix} -16 & -4 & 16 \\ 19 & -4 & -5 \\ 2 & 4 & -2 \end{bmatrix}$

**Problem 15.17.** *Using just the record of the row operations in Problem 15.16 compute the determinant of each matrix.*

**(a) Solution:** Looking at the effects of the row operations on the $\det(A)$ we get

$R_2 = R_2 - 2R_1$, $R_3 R_3 + 6R_1$: leaves determinant unchanged.

$R_3 = R_3 - 2R_2$: leaves determinant unchanged.

$R_3 = R_3/(-2)$: multiplies the determinant by -1/2.

Since $\det(I) = 1$ this gives us $(-1/2)\det(A) = 1$, so $\det(A) = -2$.

**(b) Solution:** $\det(B) = 0$, because determinant of row reduced form is 0.

**(c) Solution:** No determinant: the matrix is not square.

**(d) Solution:** The only row operations that change the determinant are $R_2 = R_2/(-2)$ and $R_3 = R_3/(-14)$. So $(-1/2)(-1/14)\det(D) = 1 \quad \Rightarrow \det(D) = 28$.

**Problem 15.18.** *Compute the transpose of the following matrices.*

$$A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}. \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

**Solution:** $A^T = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad C^T = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad D^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

**Problem 15.19.** *Let* $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

*Show by direct computation that* $(AD)^T = (D^T A^T)$.

**Solution:** $AD = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 31 & 42 & 53 & 64 \\ 11 & 14 & 17 & 20 \end{bmatrix}$ and

$D^T A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 31 & 11 \\ 42 & 14 \\ 53 & 17 \\ 64 & 20 \end{bmatrix}$. Now, by inspection, we see that $(AD)^T = D^T A^T$.

**Problem 15.20.** **(a)** *Recall the notation for inner product:* $\langle \mathbf{v}, \mathbf{w} \rangle$. *Assume* $\mathbf{v}$ *and* $\mathbf{w}$ *are column vectors. Write the formula for inner product in terms of transpose and matrix multiplication.*

**Solution:** $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w}$. For example

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = 23.$$

**(b)** *Using this definition show* $\langle A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A^T \mathbf{w} \rangle$.

**Solution:** $\langle A\mathbf{v}, \mathbf{w} \rangle = (A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T A^T \mathbf{w} = \langle \mathbf{v}, A^T \mathbf{w} \rangle$.

You are not responsible for orthogonal matrices. The following is just for fun!

**Problem 16.21.**

**(a)** *An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently,* $S$ *is orthogonal if* $S^{-1} = S^T$.

*Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. *Find an orthogonal matrix* $S$ *and a diagonal matrix* $\Lambda$ *such that* $A = S \Lambda S^{-1}$

**Solution:** The problem is asking us to diagonalize $A$, taking care that the matrix $S$ is orthogonal.

$A$ has characteristic equation: $\lambda^2 - 2\lambda - 3$. So it has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$. By inspection (or computation), we have eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These are clearly orthogonal to each other. We normalize their lengths and use the normalized eigenvectors in the matrix $S$.

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad \Rightarrow A = S\Lambda S^{-1}.$$

Note: $A$ is a symmetric matrix. It turns out that symmetric matrix has an orthonormal set of basic eigenvectors.

**(b)** *Decouple the equation* $\mathbf{x}' = A\mathbf{x}$, *with* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

**Solution:** The decoupling change of variable is $\mathbf{u} = S^{-1}\mathbf{x} \;\Leftrightarrow\; \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The decoupled system is $\mathbf{u}' = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} \;\Leftrightarrow\; \begin{cases} u_1' = -u_1 \\ u_2' = 3u_2 \end{cases}$.