ES.1803 Problem Set 1, Spring 2024 Solutions

Part II (77 points)

Problem 1 (Topic 1) (20: 10,10)
I have a bank account whose interest is compounded continuously with a variable interest rate $a(t)$. In 2023 I neither deposited nor withdrew money.

(a) *Using the definition of deriviative as a limit of $\frac{\Delta x}{\Delta t}$, explain why the differential equation satisfied by the amount of money, $x(t)$ is $x'(t) = a(t)x(t)$.*

Solution: When going from time t to $t + \Delta t$, we have $\Delta x =$ interest earned over the period Δt. That is,

$$\Delta x \approx a(t)x(t)\Delta t \quad \text{or} \quad \frac{\Delta x}{\Delta t} \approx a(t)x(t).$$

In the limit, the approximation becomes both exact and gives the differential equation we want.

(b) *Much to my surprise, I discovered my balance rose only linearly. That is, $x(t) = mt + b$ for certain positive constants m and b. Is this possible? I suspect foul play and decide to apply the methods of 18.03. What can I conclude about the interest rate (in terms of m and b).*

We could solve the (separable) DE and solve for $a(t)$, but a much easier method is to differentiate the given formula, $x(t) = mt + b$. This gives $x' = m$. Combining this with the DE from Part (a) we get,

$$m = a(t)x(t) = a(t)(mt + b) \quad \Rightarrow \quad a(t) = m/(mt + b).$$

So what happened is possible. Whether or not you see this as foul play depends on how you feel about a bank that continuously lowers its interest rate over time.

Problem 2 (Topic 3) (15: 10,5)
(a) *This problem is warmup for Problem 3. It is about matching initial and final conditions at transition points of the input.*

Solve the DE $x' + kx = f(t)$, $x(0) = 0$, where the input $f(t)$ is given by

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < 0.5 \\ 0 & \text{for } 0.5 \le t < 1 \\ 1 & \text{for } 1 \le t. \end{cases}$$

Give the solution $x(t)$ in cases format*. (The function $f(t)$ above is given in cases format).*

Use the format shown in the circuit example in the Topic 3 notes. For example, in the interval $0.5 \le t < 1$ the solution is $x(t) = C_1 e^{-k(t-0.5)}$, where $C_1 = x(0.5)$ matches the value of $x(t)$ at the end of the interval $0 \le t < 0.5$.

Solution: The key to this problem is organizing the coefficients. First we solve the following two DEs with arbitrary initial conditions:

$$x' + kx = 0;\ x(t_0) = b. \tag{1}$$

This is our standard exponential decay equation. We write the solution as

$$x(t) = be^{-k(t-t_0)}. \qquad \text{(Solution to 1)}$$

and

$$x' + kx = 1; \; x(t_0) = b. \qquad (2)$$

This has a constant input, so we guess a constant solution $x(t) = c$. Substituting this into the equation gives $c = 1/k$. Now, the superposition principle says that we can add the homogeneous solution to our constant solution to get the general solution: $x(t) = 1/k + Ce^{-k(t-t_0)}$. Finally, the initial condition gives

$$x(t) = \frac{1}{k} + (b - \frac{1}{k})e^{-k(t-t_0)}. \qquad \text{(Solution to 2)}$$

Now we have to patch together each of the intervals in the problem.

On $0 < t < 0.5$: the DE is Equation 2 and the IC is $x(0) = 0$. So,

$$x(t) = \frac{1}{k} - \frac{1}{k}e^{-kt}.$$

At the end of the interval we have $t = 0.5$, and

$$x(0.5) = \frac{1}{k}(1 - e^{-0.5k}) \; := x_{0.5}.$$

Note, that to avoid rewriting the messy formula we gave $x(0.5)$ the name $x_{0.5}$.

The next interval is $0.5 < t < 1$. The DE is Equation 1 and the IC comes from the previous interval: $x(0.5) = x_{0.5}$. So the solution on this interval is

$$x(t) = x_{0.5}e^{-k(t-0.5)}$$

At the end of the interval we have $t = 1$ and

$$x(1) = x_{0.5}e^{-k(0.5)} := x_1.$$

The next interval is $1 < t$. The DE is again Equation 2 and the IC is $x(1) = x_1$. So the solution on this interval is

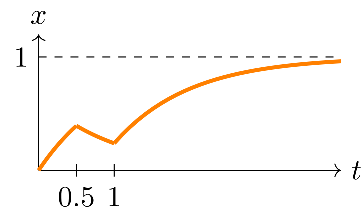
$$x(t) = \frac{1}{k} + \left(x_1 - \frac{1}{k}\right)e^{-k(t-1)}.$$

Collecting up terms: $x_{0.5} = \frac{1}{k}(1 - e^{-0.5k})$, $x_1 = x_{0.5}e^{-k(0.5)} = \frac{1}{k}(e^{-0.5k} - e^{-k})$. Here is the full solution in cases format

$$x(t) = \begin{cases} \frac{1}{k} - \frac{1}{k}e^{-kt} & \text{for } 0 < t < 0.5 \\ \frac{1}{k}\left(1 - e^{-0.5k}\right)e^{-k(t-0.5)} & \text{for } 0.5 < t < 1 \\ \frac{1}{k} + \frac{1}{k}\left(-1 + e^{-0.5k} - e^{-k}\right)e^{-k(t-1)} & \text{for } 1 < t. \end{cases}$$

(b) *For $k = 1$: graph the solution for Part (a). If this DE is modeling a mixing-tank situation, describe in words the behavior of the level of salt in the tank that this graph is showing. Aslo explain how the behavior of the response relates to the input salt rate $f(t)$.*

Solution: The amount of salt rises or falls depending on whether input is 1 or 0. The corners in the graph show the abrupt change in input. When the graph falls, it is decaying exponentially.



Problem 3 (Topic 3) (22: 6,6,10)
A population of lemmings, crazed by global warming, has been flinging themselves into the sea at a rate faster than they can reproduce. As a result the deathrate of the lemmings is now greater than the birthrate, so the population is in decline. Studies show that if nothing is done, then the lemming population is halved every 3 years.

(a) *Let x be the number of lemmings, t the time in years, $k > 0$ the decay rate of the lemming population.*

(i) *Modeling this with continuous variables, show that the DE for x as a function of t is $x' + kx = 0$. (We're just looking for a one line answer.)*

(ii) *Give the value of k, including units.*

(iii) *Assume that at time t_0 the population is x_0. Give the solution to the DE in (i) that satisfies this initial condition. (This gives the 'natural' behavior of the system, i.e., the population of lemmings, over time, when there is no input.)*

Solution: (i) The exponential decay is modeled as $x' = -kx \quad \Rightarrow x' + kx = 0$.

(ii) Since the population is halved in 3 years, $e^{-k \cdot 3 \text{ years}} = 1/2$, or $k = \dfrac{\ln(2)}{3 \text{ years}} \approx \dfrac{0.231}{\text{years}}$.

(iii) This is standard exponential decay: The solution is $x(t) = x_0 e^{-k(t-t_0)}$.

(b) *Alarmed by the potential loss of tourist business if the big annual lemming run should disappear, an importation/stocking program has been introduced by the Greenland Chamber of Commerce. For 6 months of the year (Jan. 1 to June 30), they import Canadian lemmings at a constant rate of r lemmings per year. For the other 6 months (July 1 to Dec. 31) the lemmings are left to their own devices.*

(i) *Show that $x' + kx = r$ is the DE that models the population when restocking is occurring. (Again, one line will suffice.)*

(ii) *Assume that, at time t_0, the population is x_0. Give the solution to the DE in (i) that satisfies this initial condition.*

Solution: (i) The constant input of lemmings is modeled by $x' = r$. Combining this with the exponential decay of Part (a) gives: $x' = -kx + r$ or $x' + kx = r$.

(ii) This equation is both separable and first-order linear. It is also the same equation as (2) in the solution to (2a) except with the input r instead of 1.

With initial value at t_0, it is best to write the solution as: $x(t) = r/k + (x_0 - r/k)e^{-k(t-t_0)}$.

(c) *Assume on Jan. 1 of 2023, the population was 1000. Also, assume the stocking rate* *r is 2000 lemmings per year. Using your answers to Parts (a) and (b), find the lemming population on Jan. 1, 2025. (For the purposes of this problem, you should take the period Jan. 1 to July 1 to be exactly 0.5 years and $k = 0.25$.)*

So we all get the same answer, keep the full precision for intermediate values, but report your final answer as an integer. (It doesn't pay to contemplate fractional lemmings.)

Solution: Constants and units: t in years, $k = 0.25$ /years, $r = 2000$ lemmings/year, $r/k = 8000$ lemmings.

Parts (a) and (b) give: No stocking: $x(t) = x(t_0)e^{-k(t-t_0)}$.
Stocking: $x(t) = r/k + (x(t_0) - r/k)e^{-k(t-t_0)}$.

We use these solutions to make the following table.

$0 \le t < 0.5$:	(Stocking)	$t_0 = 0$,	$x_0 = x(0) = 1000$	$\Rightarrow x(0.5) = 1822.522$.
$0.5 \le t < 1$:	(Not stocking)	$t_0 = 0.5$,	$x_0 = x(0.5) = 1822.522$	$\Rightarrow x(1) = 1608.37$.
$1 \le t < 1.5$:	(Stocking)	$t_0 = 1$,	$x_0 = x(1) = 1608.37$	$\Rightarrow x(1.5) = 2359.406$.
$1.5 \le t < 2$:	(Not stocking)	$t_0 = 1.5$,	$x_0 = x(1.5) = 2359.406$	$\Rightarrow x(2) = 2082.169$.

Our model says that on Jan. 1, 2025 ($t = 2$) there will be 2082 lemmings.

Problem 4 (Topic 1) (20: 10,10)

Read Section 1.11 (orthogonal trajectories) in the notes for Topic 1.

For each of the following families of curves,

(i) find the ODE satisfied by the family

(ii) find the orthogonal trajectories to the given family

(iii) sketch the given family and the orthogonal trajectories.

(a) *The family of curves $x = -y^2/2 + c$, where c is different for each member of the family.*

Solution: (i) Isolating the c and then differentiating with respect to x, we get, for each curve in the family

$$x + \frac{y^2}{2} = c \quad \Rightarrow 1 + y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{y}.$$

(ii) The orthogonal trajectories have derivative $\dfrac{dy}{dx} = y$.

Solving this equation (it is our most important DE) gives the orthogonal family $\boxed{y = c\,e^x}$.

(iii) Figure with (b) below.

(b) *The family of curves $x^2 + 2y^2 = c$.*

Solution: (i) Taking the derivative with respect to x, we get

$$2x + 4y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}$$

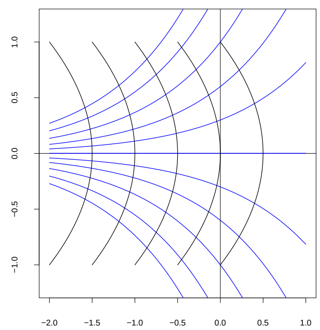
(ii) The orthogonal trajectories have derivative $\dfrac{dy}{dx} = \dfrac{2y}{x}$.

Solving this equation by separation of variables gives $\dfrac{dy}{y} = \dfrac{2dx}{x} \Rightarrow \ln|y| = 2\ln|x| + c.$

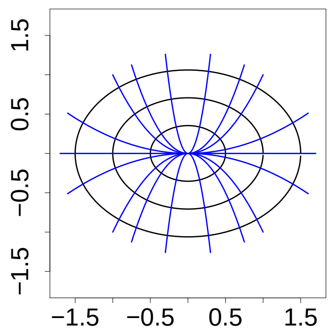
Exponentiating this gives $|y| = e^c x^2$. Dropping the absolute value, we get $y = \begin{cases} e^c x^2 & \text{if } y > 0 \\ -e^c x^2 & \text{if } y < 0. \end{cases}$

Now, remembering the lost solution $y(x) \equiv 0$, we get the full solution

$$y = cx^2, \text{ where the parameter } c \text{ can take any value.}$$



Part (a)



Part (b)

End of pset 1 solutions