ES.1803 Problem Set 2, Spring 2024

Part I (18 points)

Topic 4 (M, Feb. 12) Complex numbers and exponentials.
Read: Topic 4 notes.
Hand in: Part I problems: 4.1, 4.2b, 4.3, 4.4ab, 4.5 (posted with psets).

Topic 5 (R, Feb. 15) Linear DEs, CC homogeneous case.
Read: Topic 5 notes.
Hand in: Part I problems 5.1ab, 5.2a (posted with psets).

Part II (95 points + 5 EC points)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problem 1 (Topic 4) (15: 10,5)
For this problem you might want to first spend 5 minutes playing with the `Complex Roots` applet at https://mathlets.org/mathlets/complex-roots/

(a) For each of the following compute all three cube roots and plot them in the complex plane.

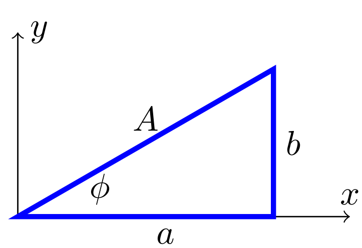
(i) -1 (ii) $1 + \sqrt{3}i$

(b) Without computation (but with the applet if you like) describe the common pattern for $(1)^{1/7}$ $(-1)^{1/7}$ and $(i)^{1/7}$. (We're looking for a short simple answer.)

Problem 2 (Topic 4) (10)
The sinusoidal identity relates a sum of sinusoids in rectangular and polar (or amplitude phase) form:

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi),$$

with a, b, A and ϕ given in the figure below



Verify this identity.

Problem 3 (Topic 5) (15: 5,5,5)
A linear, constant coefficient, homogeneous DE is called stable if all solutions go to 0 as t goes to infinity. Decide whether the following are stable. (Later we will connect this notion of stability to physical systems.)

(a) $x'' + 8x' + 7x = 0$

(b) $x'' - 9x = 0$

(c) $x'' + 2x' + 4x = 0$

Problem 4 (Topic 5) (30: 10,4,4,4,8)
Parts a-d of this problem deal with the equation $x^{(4)} - x = 0$.

(a) Give the general *real-valued* solution to the equation.

(b) Describe all the *real-valued* periodic solutions.

(c) Describe all the solutions that go to 0 as $t \to \infty$.

(d) Describe the behavior of the general solution found in Part (a) as t goes to ∞.

(e) Write down a third-order, constant coefficient, linear DE with the following properties:

(i) Its characteristic polynomial, $P(r)$ has integer coefficients.
(ii) $P(r)$ has one real and two complex roots.
(iii) All solutions of the DE tend to 0 as $t \to \infty$.

Hint: start with the roots of the characteristic polynomial.

Problem 5 (Topic 5) **Damping** (25: 5,10,10,0,0)
An important use of damping is to bring a system into equilibrium. In many mechanical systems, vibrations are a noisy nuisance or even dangerous. For example, if your car hits a bump, or wind shakes a building, or your airplane wing starts to vibrate, then you want them to settle down promptly.

Consider our standard equation modeling a damped harmonic oscillator:

$$mx'' + bx' + kx = 0$$

By dividing by m and changing the letters used, we can write this in the form

$$x'' + 2\zeta\omega x' + \omega^2 x = 0.$$

Here, ζ (Greek zeta) is called the damping ratio and ω is called the natural frequency of the oscillator.

(a) (i) Give the dimensions of ζ and ω.

(ii) Solve the equation when $\zeta = 0$. Why is ω called the natural frequency of the spring?

(b) Give the ranges of $\zeta \geq 0$ where the system is overdamped, underdamped, critically damped and undamped. Give the general real-valued solution in each case.

(c) Now we are going to use an applet to explore how quickly each type of damping reaches equilibrium. Open and play with the applet
`https://web.mit.edu/jorloff/www/OCW-ES1803/zw-compare.html`

Applet hints. 1. You can set the time by sliding the little vertical line on the t-axis or by clicking anywhere near the t-axis.

2. Once the t-slider is selected you can use the arrow keys to move it one step at a time. By using the t-zoom, you can make the step size smaller.

3. The arrow keys work on any selected slider.

4. You can drag the graph left or right.

Equilibrium is when $x = 0$ and, at least according to our model, it always takes an infinite amount of time for the system to return to equilibrium. So, as a practical solution, let's say the system has 'reached practical equilibrium' once $|x(t)|$ is permanently less than 0.005. That is, once the solution reaches and stays within a small range of $x = 0$.

Now, set $\omega = 1.0$, $b_0 = 1$, $b_1 = 0.0$ and set

$$\zeta_1 = 0.8,\ \zeta_2 = 1.0,\ \zeta_3 = 1.1,$$

Here is a strategy for finding when the system reaches practical equilibrium. Zoom out to get a sense of where each graph reaches practical equilibrium. Then set the t-zoom to 1.0. As you get close, you should have the x-zoom set to around 0.01. Select the time slider and, using the arrow keys, find the time when each damping level reaches equilibrium. In order to avoid fussing too much, you should give your answer to 1 decimal place.

For each ζ_i, find the time when the solution reaches practical equilibrium.

(d) (optional, 0 points) I love playing with this applet. So here's a challenge: With $\omega = 1.0$, $b_0 = 1.0$, $b_1 = 0.0$, find the value of the damping ratio ζ that causes x to reach practical equilibrium the fastest.

Use the zoom levels and arrow keys to find the answer to 4 or 5 decimal places

(e) (optional, 0 points) Check the 'Show roots' check box and play with the applet. This shows what is called the pole diagram for the system.

How can you tell from the pole diagram if a system is oscillatory?

How can you tell from the pole diagram if a system is stable?

Problem 6 (Extra credit) (Topic 5) (5)
In this problem we consider the law of conservation of energy in a simple harmonic oscillator. We will show that the differential equation is consistent with this law. To be concrete, let's think about an undamped spring-mass system. Suppose we have mass m and spring constant k, then the DE modeling this system is

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{1}$$

Using your 8.01 knowledge, write the total energy of the system as kinetic + potential energy. Then use the DE to show that the total energy in the system is constant.

End of pset 2