ES.1803 Problem Set 3, Spring 2024 Solutions

Part II (104 + 5 EC points)

Problem 1 (Topic 6) (10: 5,5)
Let $P(D) = D^4 + 3D^3 + 3D^2 + 3D + 2I$.

(a) *Find a particular solution to $P(D)x = e^{2t}$.*

Solution: Particular solution (using ERF): The characteristic polyomial is $P(r) = r^4 + 3r^3 + 3r^2 + 3r + 2$, so $P(2) = 60$. The ERF now gives $\boxed{x_p(t) = \frac{e^{2t}}{P(2)} = \frac{e^{2t}}{60}}$.

(b) *Find a particular solution to $P(D)x = e^{-2t}$.*

Solution: Again, $P(r) = r^4 + 3r^3 + 3r^2 + 3r + 2 \quad \Rightarrow P(-2) = 0$.

So we need the extended ERF:

$$P'(r) = 4r^3 + 9r^2 + 6r + 3 \quad \Rightarrow P(-2) = -5 \quad \Rightarrow \boxed{x_p(t) = \frac{te^{-2t}}{P'(-2)} = -\frac{te^{-2t}}{5}.}$$

Problem 2 (Topic 6) (24: 8, 8, 8)
This is the start of a problem which we will finish in Pset 4, once we have the engineering language from Topic 9 at our disposal.

We will look at the following three DEs modeling a damped spring-mass system driven in various ways.

(i) $mx'' + bx' + kx = kf(t)$ (system driven through the spring)

(ii) $mx'' + bx' + kx = bf'(t)$ (system driven through the dashpot)

(iii) $mx'' + bx' + kx = m\omega^2 f(t)$ (system driven by an unbalanced flywheel).

As usual, m, b and k are constants. The physical systems and the explanation of the models are given for (i) and (ii) in the Topic 6 notes. The system in (iii) is taken from the textbook by Edwards and Penney Section 2.6, Problem 28.

For all parts of this problem, let $f(t) = B\cos(\omega t)$ and $P(D) = mD^2 + bD + kI$.

(a) *Find the periodic solution to $P(D)x = kf(t)$. Write your answer both formally in terms of $|P(i\omega)|$ and $\mathrm{Arg}(P(i\omega))$ and in detail, in terms of m, b, k, and ω.*

Compute $P(iw) = k - m\omega^2 + bi\omega$. So,

$$|P(i\omega)| = \sqrt{(k - m\omega^2)^2 + b^2\omega^2} \text{ and } \boxed{\phi(\omega) = \mathrm{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right) \text{ in Q1 or Q2.}}$$

Thus the SRF gives: $\boxed{x_p(t) = \frac{Bk}{|P(i\omega)|}\cos(\omega t - \phi(\omega)) = \frac{Bk}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi(\omega)).}$

(b) *Repeat Part (a) for the DE: $P(D)x = bf'(t)$. For this problem, we want the solution to use cos and not sin, so we can readily see the phase lag. One way to do this is to use complex replaceement before taking the derivative f'.*

Following the suggestion, we complexify before taking the derivative: (Also remember that $i = e^{i\pi/2}$.)

$$P(D)z = Bb\left(e^{i\omega t}\right)' = iBb\omega e^{i\omega t}, \quad x = \mathrm{Re}(z).$$

ERF.:

$$z_p(t) = \frac{iBb\omega e^{i\omega t}}{P(i\omega)} = \frac{e^{i\pi/2}Bb\omega e^{i\omega t}}{|P(i\omega)|e^{i\phi_1(\omega)}},$$

where $\phi_1(\omega) = \mathrm{Arg}(P(i\omega))$. (We use a new symbol ϕ_1, because we want to reserve ϕ for the phase lag.) Taking the real part we have:

$$\boxed{x_p(t) = \frac{Bb\omega}{|P(i\omega)|}\cos(\omega t + \pi/2 - \phi_1(\omega)) = \frac{Bb\omega}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi(\omega))},$$

where $\boxed{\phi(\omega) = \mathrm{Arg}(P(i\omega)) - \frac{\pi}{2}.}$ $\mathrm{Arg}(P(i\omega))$ is the same as in Part (a).

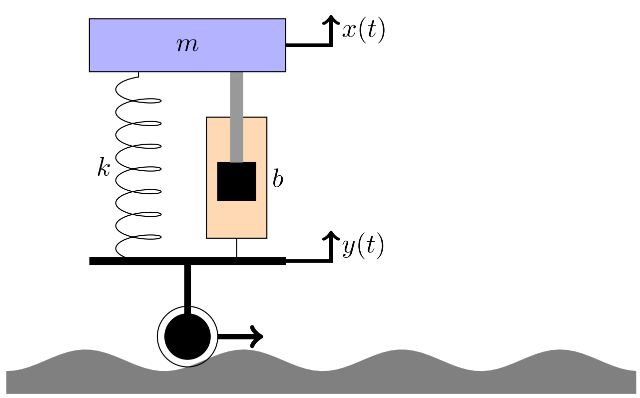
(c) *Repeat Part (a) for the DE:* $P(D)x = m\omega^2 f(t)$.

Since $m\omega^2$ is a constant, we can resuse the solution to Part (a) (adjusting the constant factor).

$$\boxed{x_p(t) = \frac{Bm\omega^2}{|P(i\omega)|}\cos(\omega t - \phi(\omega)) = \frac{Bm\omega^2}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi(\omega))\,,}$$

where $\boxed{\phi(\omega) = \mathrm{Arg}(P(i\omega)).}$ $\mathrm{Arg}(P(i\omega))$ is the same as in Part (a).

Problem 3 (Topic 6) (20: 10,10)
In this problem we will model a car running over an uneven surface. To simplify the picture we'll assume the car has just one wheel as shown in the figure.



(a) *As the wheel rolls over the road, the height $y(t)$ changes. This pushes on the suspension system, which consists of a spring with spring constant k and a damper with damping constant b. Assuming the car has mass m (and the other components are massless), find a DE to model the vertical motion $x(t)$ of the car.*

So we all get the same equation, let $x = 0$, $y = 0$ be a configuration where the spring is relaxed. So you will have to include the force of gravity in your model.

Solution: This is very similar to the examples at the end of the Topic 6 notes. We take $x = 0$, $y = 0$ be a configuration where the spring is relaxed. We have the following forces on the mass:
spring force: $-k(x - y)$
damping force: $-b(\dot{x} - \dot{y})$
gravitational force: $-mg$. So the model is

$$m\ddot{x} = -k(x-y) - b(\dot{x} - \dot{y}) - mg \Leftrightarrow \boxed{m\ddot{x} + b\dot{x} + kx = ky + b\dot{y} - mg.}$$

(Another good choice of coordinates is have $x = 0$, $y = 0$ when the spring is compressed enough to exactly hold the weight of the car. With these coordinates, the equation is the same as above without the mg term.)

(b) *Let $y = B\cos(\omega t)$, where $B = 0.4$, $\omega = 2$.*

Suppose $m = 10^3$ kg, $k = 10^5$ kg/sec^2, $b = 5 \cdot 10^3$ kg/sec. Find the periodic solution to your DE from Part (a).

Solution: I find the long decimals annoying, so I'll work with symbols and only substitute numbers at the end. There are three terms on the right side of the DE. We'll solve one term at a time and get the full solution by superposition. The three equations are

$$\begin{aligned} m\ddot{x}_1 + b\dot{x}_1 + kx_1 &= ky, \\ m\ddot{x}_2 + b\dot{x}_2 + kx_2 &= b\dot{y}, \\ m\ddot{x}_3 + b\dot{x}_3 + kx_3 &= -mg. \end{aligned}$$

Problem 2(a) tells us $x_{1,p}(t) = \dfrac{Bk}{|P(i\omega)|}\cos(\omega t - \phi_1(\omega))$, where $\phi_1(\omega) = \text{Arg}(P(i\omega))$.

Problem 2(b) tells us $x_{2,p}(t) = \dfrac{Bb\omega}{|P(i\omega)|}\cos(\omega t + \pi/2 - \phi_1(\omega))$, where $\phi_1(\omega) = \text{Arg}(P(i\omega))$.

Guessing a constant solution, we get $x_{3,p}(t) = -\dfrac{mg}{k}$.

Substituting numbers: $m = 10^3$, $k = 10^5$, $b = 5 \cdot 10^3$, $B = 0.4$, $\omega = 2$, we get

$$P(i\omega) = k - m\omega^2 + ib\omega = (10^5 - 4 \cdot 10^3) + 10^4\, i.$$

So, $\boxed{|P(i\omega)| = \sqrt{(10^5 - 4 \cdot 10^3)^2 + 10^8} = 96519.43}$ and (to a ridiculous precision)

$\boxed{\phi_1(\omega) = \text{Arg}(P(i\omega) = \tan^{-1}(10^4/(10^5 - 4 \cdot 10^3)) = 0.1037923}$.

Summing we get

$$x_p(t) = x_{1,p}(t) + x_{2,p}(t) + x_{3,p}(t) = \boxed{\frac{Bk}{|P(i\omega)|}\cos(\omega t - \phi_1(\omega)) + \frac{Bb\omega}{|P(i\omega)|}\cos(\omega t + \pi/2 - \phi_1(\omega)) - \frac{mg}{k}}.$$

With the boxed expressions for $|P(i\omega)|$ and $\phi_1(\omega)$, this is the solution to the problem.

To use the problem set checker, we used a calculator to compute numerical coefficients:

$$x_p(t) = 0.4144243\cos(2t - 0.1037923) + 0.04144243\cos(2t + \pi/2 - 0.1037923) - 0.098$$

Note: Another approach is to complexify the equation

$$m\ddot{z} + b\dot{z} + kz = (k + ib\omega)e^{i\omega t} - mg, \quad x = \mathrm{Re}(z).$$

Using the ERF and putting coefficients in polar form gives

$$z_p(t) = \frac{|k+ib\omega|e^{i(\omega t + \phi_2(\omega) - \phi_1(\omega))}}{|P(i\omega)|} - \frac{mg}{k}, \text{ where } \phi_2(\omega) = \mathrm{Arg}(k+ib\omega),\ \phi_1(\omega) = \mathrm{Arg}(P(i\omega)).$$

Taking the real part, $x_p(t) = \dfrac{|k+ib\omega|}{|P(i\omega)|}\cos(\omega t + \phi_2(\omega) - \phi_1(\omega)) - \dfrac{mg}{k}$. This method has the advantage of giving the answer directly in amplitude-phase form.

Problem 4 (Topic 7) (10 + 5EC)

(a) *Let* $\mathrm{D} = \dfrac{d}{dt}$. *Show that* $(\mathrm{D} + tI)(\mathrm{D} - tI) = \mathrm{D}^2 - (1 + t^2)I$.

We apply $(D+t)(D-t)$ to a *test* function y to see what it does.

$$\begin{aligned}(D+tI)(D-tI)y &= (D+tI)(y' - ty)\\ &= D(y'-ty) + t(y'-ty)\\ &= y'' - y - ty' + ty' - t^2y\\ &= y'' - (1+t^2)y\\ &= (D^2 - (1+t^2)I)y.\end{aligned}$$

This is what we wanted to show.

(b) (Extra credit) *Use the result of Part (a) to find the general solution to the non-constant coefficient, second-order, linear DE* $y'' - (1+t^2)y = 0$. *(One of your two independent solutions will need to be left as an integral.)*

We use the factored form of the equation: $(D+t)(D-t)y = 0$

We solve in two steps. First, let $v = (D-t)y$ and solve $(D+t)v = 0$.

This becomes $v' + tv = 0$. It is separable and gives $v = C_1e^{-t^2/2}$.

Next solve $(D-t)y = v \Leftrightarrow v' - tv = C_1e^{-t^2/2}$.

This is also a linear equation. Using the definite integral version of the variation of parameters method for solving first-order linear equations, we get

$$y = e^{t^2/2}(C_2 + \int_0^t C_1e^{-u^2}\,du).$$

Problem 5 (Topic 8) (20: 10, 5, 5)

Let $P(r) = r^5 + 5r^4 + 13r^3 + 19r^2 + 19r + 16$.

(a) *Use a numerical solver, e.g., Wolfram Alpha, to find the roots of* $P(r)$.

Solution: Using Wolfram Alpha, I get roots

$$-1.88186,\ -1.50084 \pm 1.69421i,\ -0.0582301 \pm 1.28696i.$$

(b) *Is the system* $P(D)x = f(t)$ *stable?*

Solution: Yes, all the roots have negative real part.

(c) *Give the general real-valued solution to $P(D)x = 2$.*

Solution: The DE is $x^{(5)} + 5x^{(4)} + 13x''' + 19x'' + 19x' + 16x = 2$.

Rather than rewrite the numerical values of the roots in Part (a), let's give them names:

$$r_1 = -1.88186,\ \alpha_1 \pm \beta_1 = -1.50084 \pm 1.69421i,\ \alpha_2 \pm \beta_2 = -0.0582301 \pm 1.28696i.$$

So the general real-valued homogeneous solution is

$$\boxed{x_h(t) = c_1 e^{r_1 t} + c_2 e^{\alpha_1 t}\cos(\beta_1 t) + c_3 e^{\alpha_1 t}\sin(\beta_1 t) + c_4 e^{\alpha_2 t}\cos(\beta_2 t) + c_5 e^{\alpha_2 t}\sin(\beta_2 t)}$$

For a particular solution, we try one of the form $x = C$ (C a constant). Plugging this into the DE we find $\boxed{x_p(t) = 1/8.}$

The general solution is $\boxed{x(t) = x_p(t) + x_h(t).}$

Problem 6 (Topic 6) (20: 10, 5, 5)

(a) *Show that the DE $x'' + 9x = \cos(\omega t)$ has solution*

$$x_p(t) = \begin{cases} \cos(\omega t)/|9-\omega^2| & \text{if } \omega < 3 \\ t\cos(3t - \pi/2)/6 & \text{if } \omega = 3 \\ \cos(\omega t - \pi)/|9-\omega^2| & \text{if } \omega > 3. \end{cases}$$

Solution: Preparing for the SRF: $P(i\omega) = 9 - \omega^2$. $P'(i\omega) = 2i\omega$. So,

$$\text{If } \omega \neq 3\text{: } |P(i\omega)| = |9-\omega^2|, \quad \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } \omega < 3 \\ \pi & \text{if } \omega > 3. \end{cases}$$

$$\text{If } \omega = 3\text{: } |P(i\omega)| = 0,\ P'(i\omega) = 6i \Rightarrow |P'(i\omega)| = 6, \text{Arg}(P'(i\omega)) = \pi/2.$$

So using the SRF, we get

$$\text{If } \omega \neq 3\text{: } x_p(t) = \frac{\cos(\omega t - \phi(\omega))}{|P(i\omega)|}, \text{where } \phi(\omega) = \text{Arg}(P(i\omega))$$

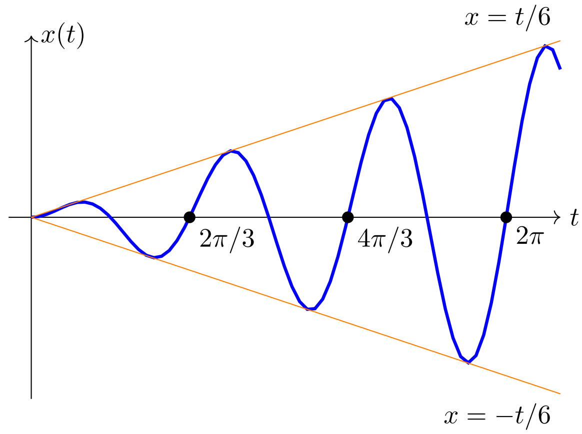
$$\Rightarrow x_p(t) = \begin{cases} \cos(\omega t)/|9-\omega^2| & \text{if } \omega < 3 \\ \cos(\omega t - \pi)/|9-\omega^2| & \text{if } \omega > 3. \end{cases}$$

$$\text{If } \omega = 3\text{: } x_p(t) = \frac{t\cos(3t - \phi(3))}{|P'(3i)|}, \text{where } \phi(3) = \text{Arg}(P'(3i)) \Rightarrow x_p(t) = \frac{t\cos(3t - \pi/2)}{6}.$$

This is what we needed to show.

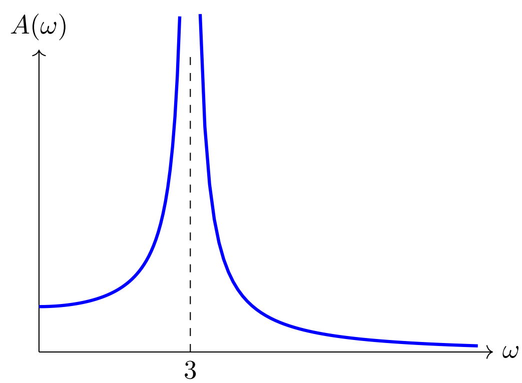
(b) *Graph the solution when $\omega = 3$.*

Solution:



(c) *The amplitude* $A = \dfrac{1}{|9 - \omega^2|}$ *is an important function of the input frequency* ω*. Graph the amplitude function.*

Solution: There is a vertical asymptote at $\omega = 3$.



End of pset 3 solutions