ES.1803 Problem Set 3, Spring 2024

Part I (30 points)

Topic 6 (R, Feb. 22) Operators, inhomogeneous DEs, exponential response formula.
Read: Topic 6 notes.
Hand in: Part I problems: 6.1 - 6.5 (posted with psets).

Topic 7 (M, Feb. 26) Inhomogeneous DEs: UC methods; Theory.
Read: Topic 7 notes.
Hand in: Part I problems 7.1abc, 7.2ab (posted with psets).

Topic 8 (T, Feb. 27) Applications: stability.
Read: Topic 8 notes.
Hand in: Part I problems: 8.1, 8.2, 8.3ab, 8.4 (posted with psets).

Part II (104 + 5 EC points)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problem 1 (Topic 6) (10: 5,5)
Let $P(D) = D^4 + 3D^3 + 3D^2 + 3D + 2I$.

(a) Find a particular solution to $P(D)x = e^{2t}$.

(b) Find a particular solution to $P(D)x = e^{-2t}$.

Problem 2 (Topic 6) (24: 8, 8, 8)
This is the start of a problem which we will finish in Pset 4, once we have the engineering language from Topic 9 at our disposal.

We will look at the following three DEs modeling a damped spring-mass system driven in various ways.

(i) $mx'' + bx' + kx = kf(t)$ (system driven through the spring)

(ii) $mx'' + bx' + kx = bf'(t)$ (system driven through the dashpot)

(iii) $mx'' + bx' + kx = m\omega^2 f(t)$ (system driven by an unbalanced flywheel).

As usual, m, b and k are constants. The physical systems and the explanation of the models are given for (i) and (ii) in the Topic 6 notes. The system in (iii) is taken from the textbook by Edwards and Penney Section 2.6, Problem 28.

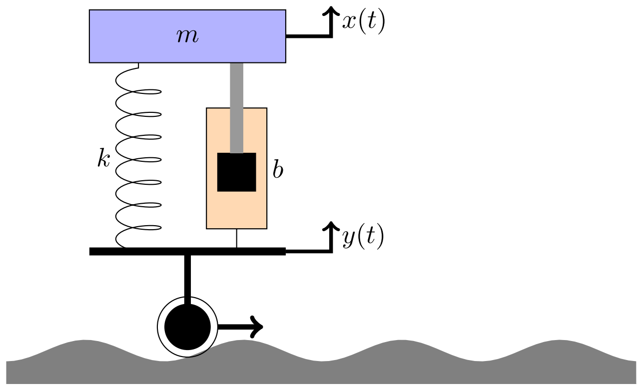
For all parts of this problem, let $f(t) = B\cos(\omega t)$ **and** $P(D) = mD^2 + bD + kI$**.**

(a) Find the periodic solution to $P(D)x = kf(t)$. Write your answer both formally in terms of $|P(i\omega)|$ and $\text{Arg}(P(i\omega))$ and in detail, in terms of m, b, k, and ω.

(b) Repeat Part (a) for the DE: $P(D)x = bf'(t)$. For this problem, we want the solution to use cos and not sin, so we can readily see the phase lag. One way to do this is to use complex replaceement before taking the derivative f'.

(c) Repeat Part (a) for the DE: $P(D)x = m\omega^2 f(t)$.

Problem 3 (Topic 6) (20: 10,10)
In this problem we will model a car running over an uneven surface. To simplify the picture we'll assume the car has just one wheel as shown in the figure.



(a) As the wheel rolls over the road, the height $y(t)$ changes. This pushes on the suspension system, which consists of a spring with spring constant k and a damper with damping constant b. Assuming the car has mass m (and the other components are massless), find a DE to model the vertical motion $x(t)$ of the car.

So we all get the same equation, let $x = 0$, $y = 0$ be a configuration where the spring is relaxed. So you will have to include the force of gravity in your model.

(b) Let $y = B\cos(\omega t)$, where $B = 0.4$, $\omega = 2$.

Suppose $m = 10^3$ kg, $k = 10^5$ kg/sec^2, $b = 5 \cdot 10^3$ kg/sec. Find the periodic solution to your DE from Part (a).

Problem 4 (Topic 7) (10 + 5EC)

(a) Let $\mathrm{D} = \dfrac{d}{dt}$. Show that $(\mathrm{D} + tI)(\mathrm{D} - tI) = \mathrm{D}^2 - (1+t^2)I$.

(b) (Extra credit) Use the result of Part (a) to find the general solution to the non-constant coefficient, second-order, linear DE $y'' - (1+t^2)y = 0$. (One of your two independent solutions will need to be left as an integral.)

Problem 5 (Topic 8) (20: 10, 5, 5)
Let $P(r) = r^5 + 5r^4 + 13r^3 + 19r^2 + 19r + 16$.

(a) Use a numerical solver, e.g., Wolfram Alpha, to find the roots of $P(r)$.

(b) Is the system $P(D)x = f(t)$ stable?

(c) Give the general real-valued solution to $P(D)x = 2$.

Problem 6 (Topic 6) (20: 10, 5, 5)

(a) Show that the DE $x'' + 9x = \cos(\omega t)$ has solution

$$x_p(t) = \begin{cases} \cos(\omega t)/|9-\omega^2| & \text{if } \omega < 3 \\ t\cos(3t - \pi/2)/6 & \text{if } \omega = 3 \\ \cos(\omega t - \pi)/|9-\omega^2| & \text{if } \omega > 3. \end{cases}$$

(b) Graph the solution when $\omega = 3$.

(c) The amplitude $A = \dfrac{1}{|9-\omega^2|}$ is an important function of the input frequency ω. Graph the amplitude function.

End of pset 3