ES.1803 Problem Set 4, Spring 2024

Part I (8 points)

Topic 9 (R, Feb. 29) Applications: frequency response.
Read: Topic 9 notes.
Hand in: Part I problems 9.1a, 8.3c, 9.3ab (posted with psets).

Part II (110 + 10 EC points)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problems 1-5 are really one long problem, aimed at filling out and understanding the table below. We will look at the following three DEs, modeling a damped spring-mass system driven in various ways. Our main focus will be on the amplitude response (or gain) and how it differs between the three systems.

System (i) $mx'' + bx' + kx = kf(t)$ (system driven through the spring)

System (ii) $mx'' + bx' + kx = bf'(t)$ (system driven through the dashpot)

System (iii) $mx'' + bx' + kx = m\omega^2 f(t)$ (system driven by an unbalanced flywheel).

In all cases the input is $f(t) = B\cos(\omega t)$, where B and ω are constants.

As usual, m, b and k are constants. The physical systems and the explanation of the models are given for (i) and (ii) in the Topic 6 notes. The system in (iii) is taken from the textbook by Edwards and Penney Section 2.6 (right before and in Problem 28).

Problem 1 (Topic 9) (15) **Output, Gain and Phase Lag**
As in Pset 3, Problem 2, let $f(t) = B\cos(\omega t)$ and $P(s) = ms^2 + bs + k$.

Either solve the DEs again or use your answers to Pset 3, Problem 2 to fill in the gain and phase lag rows in the table. For the gain, write your answer both formally, in terms of $|P(i\omega)|$, and in detail, in terms of m, b, k, and ω. For the phase lag, you just need to give the formal version in terms of $\text{Arg}(P(i\omega))$. (But be careful, not every phase lag is simply $\text{Arg}(P(i\omega))$.)

Show your work on your paper and either print out the chart or make a copy and put the final answer in that.

Now fill in the output row of the table. To make things simpler, you only need to write $x(t)$ in terms of $g(\omega)$ and $\phi(\omega)$.

	System (i)	System (ii)	System (iii)
Right-hand side of DE (Input $= f(t)$)	$kf(t)$	$bf'(t)$	$m\omega^2 f(t)$
Output $x(t)$			
Gain $g(\omega)$			
Phase lag $\phi(\omega)$			
ω_r			
Filter type	Low pass	Band pass	High pass

Table for Problems 1-5

Problem 2 (Topic 9) (30: 10,10,10) **Practical Resonance**
Hint: In all three parts of this problem, the maximum of $g(\omega)$ is at the same point as the minumum of $1/g(\omega)^2$, but the latter is easier to work with. Parts (a) and (c) require a small amount of calculus. Part (b) can be done with or without calculus.

(a) Find the practical resonant frequency ω_r for System (i). Be sure to note the conditions when there is no resonance. Add ω_r to the table.

(b) Repeat Part (a) for System (ii).

(c) Repeat Part (a) for System (iii).

Problem 3 (Topic 9) (15: 10,5) **Visualization**
(a) Start the applet: `https://mathlets.org/mathlets/amplitude-and-phase-2nd-order/`. This is a visualization of System (i). Turn on the Bode and Nyquist plots. Play with the applet, be sure you can identify which plot shows the amplitude response.

Note: The equation in the applet has $m = 1$, i.e., the coefficient of x'' is 1.

Note: When adjusting a slider, once you've selected the slider, you can get fine control using the arrow keys to move it. You can leave your mouse in the gain graph window to get a readout of the gain value.

(i) Set $k = 1$ and $b = 0.5$. Use the graphs to determine if the system has a resonant frequency. If it does, give its value. Include a sketch of the amplitude response graph –no credit if you don't label the axes.

(ii) Set $k = 1$ and $b = 1.5$ and repeat Part (i)

(iii) Check your answers in (i) and (ii) against your answer to Problem 2.

(iv) Still with $k = 1$ find the value of b where practical resonance disappears. Check this with with your answer to Problem 2.

(b) Start the applet: `https://mathlets.org/mathlets/amplitude-and-phase-2nd-order-ii/`. Check the Bode plots checkbox. Describe how the amplitude response changes as b and k are varied.

Problem 4 (Topic 9) (15: 5,5,5) **Visualization and Filtering**
The systems in this pset can be considered filters. This means that they respond differently to input signals of different frequencies. For the frequencies where the gain is relatively large, we say the filter passes the frequency, for those where the gain is small, we say it stops the frequency. Here, we'll see how the power of ω in the gain affects the shape of the filter.

Start applet: `https://web.mit.edu/jorloff/www/OCW-ES1803/mbk4.html` This applet shows graphs for the system $mx'' + bx' + kx = \omega^n \cos(\omega t)$, where n is a set-able parameter. So the applet covers all the powers of ω we've seen in the previous problems.

(a) Set $n = 0$, $m = 0.5$, $b = 1$, $k = 1$. Use the applet graphs to explain the use of the term 'low pass' in the filter row of the table.

(b) Set $n = 1$, $m = 1.5$, $b = 0.8$, $k = 5$. Use the applet graphs to explain the use of the term 'band pass' in the filter row of the table. (A range of frequencies is referred to as a band.)

(c) Set $n = 2$, $m = 1$, $b = 1$, $k = 1$. Use the applet graphs to explain the use of the term 'high pass' in the filter row of the table.

Problem 5 (Topic 9) (20: 5,5,5,5) **AM Radio Tuning and LRC Circuits**
An LRC circuit can be modeled using the same DE as in system (ii). Specifically, we often want to know the voltage V_R across the resistor. This is modeled by the DE

$$LV_R'' + RV_R' + \frac{1}{C}V_R = RE'.$$

Where L = inductance in henries, R = resistance in ohms, C = capacitance in farads and E = input EMF in volts.

(a) Assume $E = E_0 \cos(\omega t)$ and use the table to give the periodic solution for V_R in amplitude-phase form.

(b) Open the applet: `https://web.mit.edu/jorloff/www/OCW-ES1803/lrc.html`. This applet models an LRC circuit. The input voltage is a superposition of sine waves. Play with the applet –be sure to learn how to vary ω_1 and ω_2 by dragging the sliders on the amplitude plot.

Describe what happens to the amplitude response plot as L, R and C are varied.

(c) An LRC circuit can be used as part of a simple AM radio tuner. In an AM radio broadcast the signal is given by $A\cos(\omega t)$ where ω is the 'carrier' frequency (between 530 and 1600 khz). To really carry information the amplitude A must vary with time –this is the amplitude modulation– but, we will ignore this right here.

A typical range of values for this simple variable capacitor AM radio tuner is $L \approx 0.5$ microhenries, R is the resistance in the wire (very small) and C is between 0.02 and 0.2 microfarads. To keep things simple, we will use different ranges, however the idea is the same.

In the LRC Filter applet, set $\omega_1 = 1$ and $w_2 = 4$. Set the input amplitudes c_1 and c_2 to 1. Find settings for L, R and C so that the system filters out the ω_2 part of the signal i.e., the output looks (a lot) like a sine wave of frequency ω_1. Give your values for L, R and C.

Note. Since the frequencies in the applet are not in the AM range, your values for L, R, C do not have to be in the same range as those in a typical variable capacitor tuner.

(d) Since lots of stations are broadcasting at once, the antenna on your radio picks up a signal which is a superposition of lots of frequencies. The job of the tuner is to filter out all but the frequency you want. That is, the filter should pass a small band of frequencies around the desired one.

Using the applet, set $L = 1$, $R = 0.5$. Now, vary C and then explain why a variable capacitor circuit could be used as an AM radio tuner.

For amusement: check the 'N term mode' box. This changes the input to a sum of N sinusoids, with N set by a slider.

Set $N = 1$, $L = 5.0$, $R = 0.32$, $C = 0.05$, $c = 0.2$ and $\omega = 2$. You should see a sinusoidal input and a sinusoidal output. Now increase N from 1 to 2 to 3, etc. The input changes as more sinusoids are added. How does this change the output? Explain this in terms of the filter indicated by the gain graph.

Problem 6 (Topic 9) (15: 5,10,0)
There is another damped spring-mass model which can be used for further comparison and contrast, namely the automobile suspension system given in a previous problem. It is equivalent to a spring-mass system which is driven through *both* the spring and dashpot

$$mx'' + bx' + kx = kf(t) + bf'(t).$$

As before, we will take the input $f(t) = B\cos(\omega t)$.

(a) Derive the formula for the amplitude response $g(\omega)$. As before, give the formal answer in terms of $|P(i\omega)|$ and $\text{Arg}(P(i\omega))$ and the detailed answer in terms of m, b, k, ω.

(b) Derive the formula for the practical resonant frequency.

Does practical resonance always occur in this case?

(c) No question here, just a suggestion to look at the MIT mathlet:

https://mathlets.org/mathlets/amplitude-and-phase-2nd-order-iii/

Problem 7 (Topic 9) (Extra credit: 10: 5,5)
The complex gain for a system is the gain for the complexified system. For example, consider the system $P(D)x = kB\cos(\omega t)$, where $B\cos(\omega t)$ is the input. Using complex replacement, this becomes $P(D)z = kBe^{i\omega t}$. We can simplify this by writing

$$P(D)z = kBe^{st},$$

where s is any complex number. Of course, when we want to solve $P(D)x = kB\cos(\omega t)$, we take $s = i\omega$. The ERF says the solution to this is $\dfrac{kBe^{st}}{P(s)}$. If we consider the input to be Be^{st}, then the complex gain is $G(s) = \dfrac{k}{P(s)}$. (This is also known as the system or transfer function.)

The zero-pole diagram for a system is drawn in the complex plane. A pole for the complex gain is a (complex) value of s where the denominator of $G(s)$ has a 0. A zero is a value of s where $G(s) = 0$.

(a) Draw a zero-pole diagram with a zero at $s = 0$ and poles at $s = -2$, $s = -1 \pm 2i$. Write down a system, specifying the input and output, that has this as its zero-pole diagram.

(b) This system has a practical resonant frequency. Indicate the approximate location of this on the pole diagram

End of pset 4