ES.1803 Problem Set 5, Spring 2024 Solutions

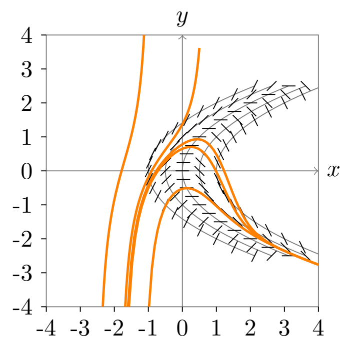
Part II (85 points)

Problem 1 (Topic 10) (10)

Consider the DE $y' = y^2 - 2x$.

Sketch the isoclines corresponding to slopes 0, 1, -1, 2, -2, and add accompanying slope field elements. Then draw in five solution curves, which illustrate the general range of behavior.

Solution: The isoclines $y^2 - 2x = m$ are sideways parabolas.



Problem 2 (Topic 10) (10)

Open the Mathlet: `https://mathlets.org/mathlets/isoclines/` *and choose the equation: $y' = (x + 2y)(x - y)$.*

Play with the applet a little. Learn how to create and position isoclines using the m-slider. Likewise for integral curves by holding down the mouse and dragging in the main graph window. Then answer the following questions.

(i) *Looking at the direction field, it seems there is a critical initial value $y(0) = K$ which determines the long-term behavior of the solutions. That is*

For $y(0) < K$, the solution $y(x)$ becomes decreasing as x increases
For $y(0) > K$, the solution $y(x)$ becomes increasing as x increases

Determine the value of K to within 0.02. Say briefly what you did to find this.

Note: the curve separating the two types of solution curves is called a separatrix. It is also a solution, the one with initial condition $y(0) = K$. Finding and understanding separatrices is a nontrivial problem.

(ii) *Let $y(x)$ be such that $y(0.5) = 0$. Estimate $y(100)$. Give a brief reason. (Hint: look at the nullcline and another isocline as fences.*

Solution: (i) $K = -0.29$. By systematically clicking at various points on the y-axis, we see the solution goes towards positive infinity for $K = -0.28$ and goes down towards minus infinity for $K = -0.30$.

(ii) Looking at the nullcline $y = x$ and the isocline with $m = 3$, we see a funnel in the upper right of the applet. (Note: since the isoclines are hyperbolas with asympotes given by the nullcline, we know the funnel width goes to 0 as x increases.) We see that the curve with initial value $y(0.5) = 0$ goes into the funnel, i.e., is asymptotic to the line $y = x$. Thus,

$y(100) \approx 100$.

Problem 3 (Topic 11) (15: 5,5,5)
In this problem the IVP is $y' = y^2 - 2x$ *with IC* $y(-0.98) = 0$.

(a) *Use your calculator and Euler's method with stepsize* $h = 0.125$ *to approximate* $y(-0.98 + \frac{k}{8})$ *for k = 0,1, 2, ..., 8. On your pset, record your results in a table. Give results to 4 decimal places.*

Solution: See below

(b) *The 'exact' values are given in the table below. Compare the results of your calculation in Part (a) with the 'exact' values given in the table. Make a table of results and errors. What would you expect to happen to the error in your approximation to* $y(0.02)$ *if you used stepsize* 0.250 *instead of* 0.125*?*

x	*y*
-0.980	0.0000
-0.855	0.2317
-0.730	0.4447
-0.605	0.6494
-0.480	0.8562
-0.355	1.0775
-0.230	1.3315
-0.105	1.6494
0.020	2.0930

Table of exact values of $y(x)$ *for the solution with initial value* $y(-0.98) = 0$.

Solution: We give one table with the answers to both (a) and (b).

(a) The Euler approximations are in the y_k column.

(b) The exact values are copied from the table given with the problem. The error column is computed as: error = approximate - exact (the absolute value would also be fine).

		Hand	*Calc*		.
k	x_k	y_k	*exact*	*error*	
0	−0.980	0.0000	0.0000	0.0000	
1	−0.855	0.2450	0.2317	0.0133	
2	−0.730	0.4663	0.4447	0.0216	
3	−0.605	0.6759	0.6494	0.0265	
4	−0.480	0.8843	0.8562	0.0281	
5	−0.355	1.1020	1.0775	0.0245	
6	−0.230	1.3426	1.3315	0.0111	
7	−0.105	1.6254	1.6494	−0.0240	
8	0.020	1.9819	2.0930	−0.1111	

If we doubled the stepsize to 0.250 we'd expect the error to roughly double.

(c) *We will use the Mathlet:* `https://mathlets.org/mathlets/eulers-method/` *to do a little more exploration of a similar equation.*

Choose $F(x, y) = y^2 - x$ *from the drop-down menu and use the mouse to set the initial condition to* $y(-0.95) = 0$. *(Initial* x *values of* -0.94 *or* -0.96 *are also fine.) Choose 'All*

Euler' and click 'Start'. Now choose 'Actual' and click 'Start'. What is going on in stepsize 1.00?

Solution: In the case of stepsize 1.0 after one step the solution has crossed a separatrix and is in a region where the integral curves go to infinity instead of rising and then falling. This is also true at stepsize 0.5. When the initial x value is -0.95, For stepizes 0.25 and 0.125 the Euler approximation stays relatively close to the actual solution.

Problem 4 (Topic 12) (20: 10,5,5)

(a) *Open the applet: `https://mathlets.org/mathlets/phase-lines/`, and check the 'Phase Line' and 'Bifurcation Diagram' checkboxes. Choose the equation $y' = ay + y^3$. Play with the slider for 'a' and watch all the plots change. (Note: the color-coding uses green for stable equilibria and red for unstable –look at the yellow arrows.)*

(i) *Give all the bifurcation point(s).*

(ii) *For each bifurcation point, how many equilibria are there on either side of the point?*

(iii) *Copy the bifurcation diagram –be sure to label your axes and indicate the stable and unstable branches of the diagram.*

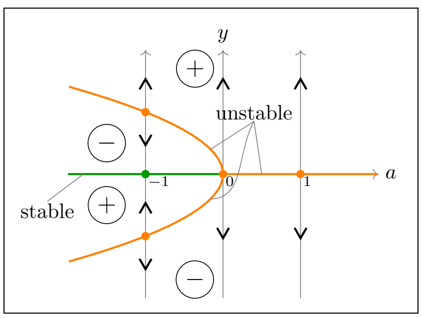
(iv) *On your bifurcation diagram, show a representative phase line for each interval (of a) determined by the bifurcation points.*

(v) *For each of the phase lines in Part (iv), sketch some representative integral curves –again, be sure to label your axes.*

Solution: (i) $a = 0$ is the only bifurcation point.

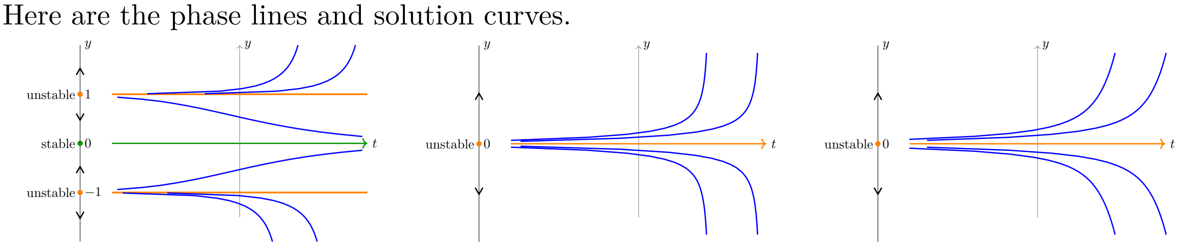
(ii) For $a > 0$ there is one equilibrium (at $y = 0$). For $a < 0$ there are three equilibria.

(iii), (iv), (v) (all copied from the applet). We add the large pluses and minuses based on the stability shown in the bifurcation diagram. (These are redundant.) We show phase lines for $a = -1, 0, 1$.



Bifurcation diagram with phase lines at $a = -1$, 0, 1

Here are the phase lines and solution curves.



$a = -1$ $a = 0$ $a = 1$

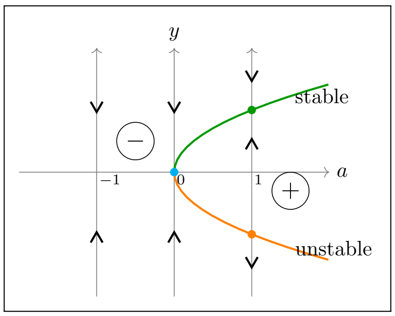
(b) *Now consider the autonomous DE with parameter a:* $y' = a - y^2$. *(This is not on the applet.)*

Draw the bifurcation diagram for this equation. Be sure to label your axes and indicate the stable and unstable branches of the diagram.

List all the bifurcation point(s). Show phase lines for each interval (of the parameter a) determined by the bifurcation points. (You can put these phase lines directly on the bifurcation diagram.)

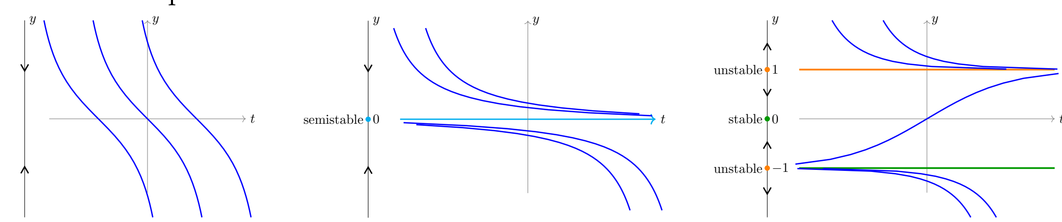
Solution: The critical points are when $y' = 0 \Leftrightarrow a = y^2$. This is a rightward opening parabola in the ay-plane.

The bifurcation diagram shows the only bifurcation point is at $a = 0$. The plot of critical points divides the plane into two regions. We mark these as plus of minus depending on the sign of y'. We include phase lines for $a = -1$, $a = 0$, $a = 1$.



Bifurcation diagram with phase lines at $a = -1$, 0, 1

Here are the phase lines and solution curves.



$a = -1$ $\qquad$ $a = 0$ $\qquad$ $a = 1$

(c) *If y in Part (b) represents a population, describe how the long-term stability of the population depends on the parameter a.*

Solution: For $a > 0$, the population stabilizes at the positive value $y = \sqrt{a}$.

For $a \le 0$, the population crashes.

Problem 5 (Topic 9) (30: 10,5,5,5,5) Review of linear constant coefficient DEs
A system is modeled by the DE $x' + kx = kf$. *Here, we'll consider f to be the input.*

(a) *Assume that* $f(t) = B\cos(\omega t)$. *Solve this linear DE to get an explicit formula for the general solution* $x = x(t)$.

Why is the term Ce^{-kt} *called the transient?*

What is the gain $g(\omega)$*?*

Solution: Homogeneous solution: $x_h(t) = Ce^{-kt}$.

Particular solution: $P(i\omega) = k+i\omega$. So, $|P(i\omega)| = \sqrt{k^2+\omega^2}$, $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{\omega}{k}\right) \text{ in Q1}}$.

So, $x_p(t) = \dfrac{kB\cos(\omega t - \phi(\omega))}{|P(i\omega)|} = \dfrac{kB\cos(\omega t - \phi(\omega))}{\sqrt{k^2+\omega^2}}$.

The general solution is $x(t) = x_p + x_h = \dfrac{kB\cos(\omega t - \phi(\omega))}{\sqrt{k^2+\omega^2}} + Ce^{-kt}$.

Ce^{-kt} is called the transient because it goes to 0 as t gets larger.

If input is $B\cos(\omega t)$, then the input amplitude is B. Since the output amplitude is $\dfrac{kB}{\sqrt{k^2+\omega^2}}$, the gain $g(\omega) = \dfrac{k}{\sqrt{k^2+\omega^2}}$.

(b) *Now suppose the input varies sinusoidally around a constant, i.e., our system is modeled by the DE*

$$x' + kx = k(R + B\cos(\omega t)).$$

(You can think of this equation as modeling the population of lemmings from Pset 1, with the additional term representing seasonal effects on the population.)

Solve this DE.

Solution: We use superposition to solve the DE in two pieces:

1. $x_1' + kx_1 = kR$, 2. $x_2' + kx_2 = kB\cos(\omega t)$.

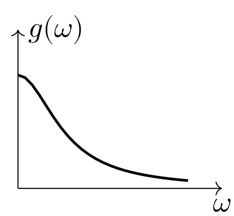
Piece 1 we can solve from memory, inspection, undetermined coefficients, exponential input theorem, separation of variables or the formula for linear first-order equations. They all give, $x_{1,p}(t) = R$.

We solved piece 2 in Part (a): $x_{2,p}(t) = \dfrac{kB\cos(\omega t - \phi(\omega))}{\sqrt{k^2+\omega^2}}$.

So the full solution is $x(t) = x_{1,p} + x_{2,p} + x_h = R + \frac{kB\cos(\omega t - \phi(\omega))}{\sqrt{k^2+\omega^2}} + Ce^{-kt}$.

(c) *Letting $B = 1$ and $k = 0.25$ plot gain as a function of ω. You can either sketch this by hand or use some tool such as Matlab to draw the graph. Hint: your graph should have g going to 0 as ω gets large.*

Solution: From Part (a): $g(\omega) = \dfrac{0.25}{\sqrt{0.25^2+\omega^2}} = \dfrac{1}{\sqrt{1+16\omega^2}}$: ($g$ is monotonically decreasing to 0.).



(d) *Describe the long-term behavior of the lemming population.*

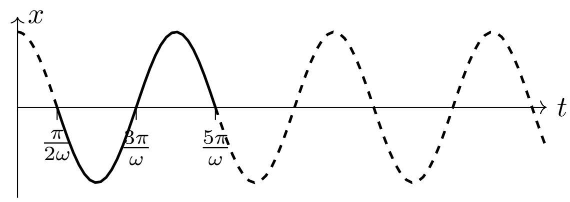
Solution: In the long-term the transient term goes to 0 and the population oscillates with an amplitude $g(\omega)B$ around the constant population R.

(e) *Suppose that the environment goes haywire and the seasonal effects happen faster and faster, so that ω becomes very large. What will the effect be on the lemming population over* *time? Give an explanation of this.*

Solution: Over time the transient goes to 0, so we'll only worry about the $R+g(\omega)B\cos(\omega t-\phi)$ part of the solution. As ω grows the formula shows the amplitude $g(\omega)B$ gets smaller. If ω becomes very large then the population will be effectively constant at R.

Noting that the gain is small is a sufficient answer. A more elaborate physical explanation has two pieces:

We don't need to do any arithmetic, but perhaps a simple computation will help. We'll look at the effect of the sinusoidal term in the input. Over one cycle from $\frac{\pi}{2\omega}$ to $\frac{5\pi}{2\omega}$.



In the first half of the cycle, from $\pi/2\omega$ to $3\pi/2\omega$, the input is negative, so lemmings are being removed. The total removed is

$$\int_{\pi/2\omega}^{3\pi/2\omega} kB\cos(\omega t)\,dt = -\frac{2kB}{\omega}. \tag{1}$$

Likewise in the second half of the cycle, between $3\pi/2$ to $5\pi/2$, the input is positive, so lemmings are being added. The total added is the same as the total removed in the first half, i.e., $\frac{2kB}{\omega}$.

(i) If ω is large the number removed/added in each half of the cycle is small. Physically this is because the cycle happens so fast that the finite rate of input doesn't have time to add very much. (You see this in the ω in the denominator in (1) and in the smaller width of the arches as ω increases.)

(ii) The same amount is gained in the second half of the cycle, so the net input is 0. Because the net of 0 happens so fast there is no time for the system to act on the changes.

Thus, (i) small numbers added and subtracted in each cycle and (ii) no effect on the system, means the system stays approximately constant around R.

End of pset 5 solutions