ES.1803 Problem Set 5, Spring 2024

Part I (20 points)

Topic 10 (R, Mar. 7) Direction fields, integral curves, existence of solutions.
Read: Topic 10 notes.
Hand in: Part I problems 10.1ab, 10.2, 10.3 (posted with psets).

Topic 11 (F, Mar. 8) Numerical methods for 1^{st} order ODEs.
Read: Topic 11 notes.
Hand in: Part I problems 11.1ab (posted with psets).

Topic 12 (M, Mar. 11) Autonomous DEs and bifurcation diagrams.
Read: Topic 12 notes.
Hand in: Part I problems 12.1abcd (posted with psets).

Part II (85 points)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

This material is visual and computational, so the pset uses a lot of applets. Please plan your time so you aren't stuck at the last minute trying to get them to work!

Problem 1 (Topic 10) (10)
Consider the DE $y' = y^2 - 2x$.

Sketch the isoclines corresponding to slopes 0, 1, -1, 2, -2, and add accompanying slope field elements. Then draw in five solution curves, which illustrate the general range of behavior.

Problem 2 (Topic 10) (10)
Open the Mathlet: https://mathlets.org/mathlets/isoclines/ and choose the equation: $y' = (x + 2y)(x - y)$.

Play with the applet a little. Learn how to create and position isoclines using the m-slider. Likewise for integral curves by holding down the mouse and dragging in the main graph window. Then answer the following questions.

(i) Looking at the direction field, it seems there is a critical initial value $y(0) = K$ which determines the long-term behavior of the solutions. That is

For $y(0) < K$, the solution $y(x)$ becomes decreasing as x increases
For $y(0) > K$, the solution $y(x)$ becomes increasing as x increases

Determine the value of K to within 0.02. Say briefly what you did to find this.

Note: the curve separating the two types of solution curves is called a separatrix. It is also a solution, the one with initial condition $y(0) = K$. Finding and understanding separatrices is a nontrivial problem.

(ii) Let $y(x)$ be such that $y(0.5) = 0$. Estimate $y(100)$. Give a brief reason. (Hint: look at the nullcline and another isocline as fences.

Problem 3 (Topic 11) (15: 5,5,5)
In this problem the IVP is $y' = y^2 - 2x$ with IC $y(-0.98) = 0$.

(a) Use your calculator and Euler's method with stepsize $h = 0.125$ to approximate $y(-0.98 + \frac{k}{8})$ for $k = 0,1, 2, ..., 8$. On your pset, record your results in a table. Give results to 4 decimal places.

(b) The 'exact' values are given in the table below. Compare the results of your calculation in Part (a) with the 'exact' values given in the table. Make a table of results and errors. What would you expect to happen to the error in your approximation to $y(0.02)$ if you used stepsize 0.250 instead of 0.125?

x	y
-0.980	0.0000
-0.855	0.2317
-0.730	0.4447
-0.605	0.6494
-0.480	0.8562
-0.355	1.0775
-0.230	1.3315
-0.105	1.6494
0.020	2.0930

Table of exact values of $y(x)$ for the solution with initial value $y(-0.98) = 0$.

(c) We will use the Mathlet: `https://mathlets.org/mathlets/eulers-method/` to do a little more exploration of a similar equation.

Choose $F(x, y) = y^2 - x$ from the drop-down menu and use the mouse to set the initial condition to $y(-0.95) = 0$. (Initial x values of -0.94 or -0.96 are also fine.) Choose 'All Euler' and click 'Start'. Now choose 'Actual' and click 'Start'. What is going on in stepsize 1.00?

Problem 4 (Topic 12) (20: 10,5,5)
(a) Open the applet: `https://mathlets.org/mathlets/phase-lines/`, and check the 'Phase Line' and 'Bifurcation Diagram' checkboxes. Choose the equation $y' = ay + y^3$. Play with the slider for 'a' and watch all the plots change. (Note: the color-coding uses green for stable equilibria and red for unstable –look at the yellow arrows.)

(i) Give all the bifurcation point(s).

(ii) For each bifurcation point, how many equilibria are there on either side of the point?

(iii) Copy the bifurcation diagram –be sure to label your axes and indicate the stable and unstable branches of the diagram.

(iv) On your bifurcation diagram, show a representative phase line for each interval (of a) determined by the bifurcation points.

(v) For each of the phase lines in Part (iv), sketch some representative integral curves –again, be sure to label your axes.

(b) Now consider the autonomous DE with parameter a: $y' = a - y^2$. (This is not on the applet.)

Draw the bifurcation diagram for this equation. Be sure to label your axes and indicate the stable and unstable branches of the diagram.

List all the bifurcation point(s). Show phase lines for each interval (of the parameter a) determined by the bifurcation points. (You can put these phase lines directly on the bifurcation diagram.)

(c) If y in Part (b) represents a population, describe how the long-term stability of the population depends on the parameter a.

Problem 5 (Topic 9) (30: 10,5,5,5,5) Review of linear constant coefficient DEs
A system is modeled by the DE $x' + kx = kf$. Here, we'll consider f to be the input.

(a) Assume that $f(t) = B\cos(\omega t)$. Solve this linear DE to get an explicit formula for the general solution $x = x(t)$.

Why is the term Ce^{-kt} called the transient?

What is the gain $g(\omega)$?

(b) Now suppose the input varies sinusoidally around a constant, i.e., our system is modeled by the DE

$$x' + kx = k(R + B\cos(\omega t)).$$

(You can think of this equation as modeling the population of lemmings from Pset 1, with the additional term representing seasonal effects on the population.)

Solve this DE.

(c) Letting $B = 1$ and $k = 0.25$ plot gain as a function of ω. You can either sketch this by hand or use some tool such as Matlab to draw the graph. Hint: your graph should have g going to 0 as ω gets large.

(d) Describe the long-term behavior of the lemming population.

(e) Suppose that the environment goes haywire and the seasonal effects happen faster and faster, so that ω becomes very large. What will the effect be on the lemming population over time? Give an explanation of this.

End of pset 5