ES.1803 Problem Set 7, Spring 2024

Part I (22 points)

Topic 20 (F, Apr. 5) Step and delta functions.
Read: Topic 20 notes.
Hand in: Part I problems 20.1-3 (posted with psets).

Topic 21 (R, Apr. 11) Fourier series: basics.
Read: Topic 21 notes.
Hand in: Part I problems 21.1abc, 2 (posted with psets).

Topic 22 (T, Apr. 12) Fourier series introduction: continued.
Read: Topic 22 notes.
Hand in: Part I problems 22.1, 22.2 (posted with psets).

Topic 23 (R, Apr. 14) Sine and cosine series; calculation tricks.
Read: Topic 23 notes.
Hand in: Part I problems: 23.1ab (posted with psets).

Part II (113 + 10 extra-credit points)

Problem 1 (Topic 20) (10)
Solve the initial value problem

$$2x''' + 12x'' + 22x' + 12x = \delta(t) + u(t)e^t, \text{ with rest initial conditions.}$$

Here $u(t)$ is the unit step function. This means that $u(t)e^t = \begin{cases} 0 & \text{for } t < 0 \\ e^t & \text{for } t > 0 \end{cases}$.

Hint: the characteristic roots are small negative integers.

Problem 2 (Topic 20) (20: 5,5,5,5) **Lemmings really are adorable**
Back in your impulsive youth you helped a population of lemmings avoid extinction. But your methods led to some tedious differential equations that no one liked solving. Now that you're older and wiser and truly understand impulses, you are ready to help the lemmings again.

Image from Wikimedia, in the public domain. Also see Wikipedia: Lemming

Recall that the population y of lemmings is modeled by

$$y' + ky = f(t),$$

where t is measured in years, $k = 1.0$ is the growth rate and $f(t)$ is the input function.

(a) Your youthful input function could be described as a periodic box.

$$f_h(t) = \begin{cases} \frac{1}{h} & \text{if } 0 < t < h \\ 0 & \text{if } h < t < 1 \\ \frac{1}{h} & \text{if } 1 < t < 1+h \\ 0 & \text{if } 1+h < t < 2 \\ \dots . \end{cases}$$

(i) Graph this function for $h = 1/2$.

(ii) How many units of lemmings were added in each yearly cycle?

(b) One problem with the input in Part (a) is that you had to spend half the year on the tundra. Another was that solving the DE with $f_h(t)$ as input took about half a year and almost made you quit ES.1803. So you decide to let h go to 0. That is, every year on January 1 you'll bring a truckload of lemmings (1 truckload = 1 unit of lemmings) to the wildlife reserve and release them all at once. (You may choose to stay a while and hope the Northern Lights are visible.)

Call the new input function $f_I(t)$. Give its formula in terms of delta functions and sketch its graph. (Let $t = 0$ be January 1, 2023.)

(c) In your absence the lemming population dwindled to nothing. So, when you started inputting lemmings, the intial population was 0. Solve the DE $y' + ky = f_I(t)$ with rest initial conditions.

(d) Now we'll look at this graphically using a mathlet. Open `https://mathlets.org/mathlets/periodic-box/`.

As usual, start the applet and familiarize yourself with its controls. Set $k = 1$ and $h = 0.5$.

(i) What happens to the response from rest as h goes to 0?

(ii) What happens to the impulse train response as t gets large?

Problem 3 (Topic 21) (10: 5,5)
Without computing any integrals give the Fourier series for the following

(a) $f(t) = \sin(t - \pi/4)$ (period 2π).

(b) $f(t) = sq(t - \pi/2)$, where $sq(t)$ is the odd square wave of period 2π and amplitude 1. (You should use the known series for $sq(t)$.)

Problem 4 (Topic 21) (28: 5,5,3,5,7,3)

(a) Let $f(t)$ be the period 2 square wave, where, over one period, $f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \end{cases}$

Taking the jumps into account, compute the generalized derivative $h(t) = f'(t)$.

(b) Starting from the Fourier series for $f(t)$, find the Fourier series for $h(t)$.

(c) On the same axes, sketch the graphs of $h(t)$ and the first term in its Fourier series.

(d) Compute the coefficients of the Fourier series of $h(t)$ directly using the integral formulas and show they are the same as in the series found in Part (b).

(e) Solve $Lx = x' + 9x = h(t)$.

(f) Does the solution in Part (e) contain any near resonant terms?

Problem 5 (Topic 22) (15: 5,5,5)

Let $f(t) = \begin{cases} 2t^2 & \text{if } 0 \le t < \frac{1}{2} \\ 1-t & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{if } 1 \le t < 2. \end{cases}$

For each of the following, sketch the graph of its Fourier series over 3 full periods.

(a) $\tilde{f}_e(t)$, the *even* periodic extension of $f(t)$ with period 4.

(b) $\tilde{f}_o(t)$, the *odd* periodic extension of $f(t)$ with period 4.

(c) $\tilde{f}(t)$, the periodic extension of $f(t)$ with period 2.

Problem 6 (Topic 22) (30) See the Fourier Sound lab exercise posted alongside this pset.

Extra credit 1 (Topic 17) (10)
(There isn't space for this problem. Since we don't want to interrupt the story, we've made it an extra credit problem.)

Continuing the story of Armand and Babette: Armand and Babette, exhausted from their seemingly endless cycles of attraction and repulsion, decide nevertheless to give it one more try, and so are off to Armadillo couples therapy. The Therapist admits that, given their current emotional patterns (which will be impossible to change quickly), it doesn't look good for a long-term stable happy relationship, but suggests a short-term external intervention to see if a break in the pattern might give them some time to work on the deeper issues. The Therapist therefore sends them to the Wizard, who concocts a special potion for them. Again let $x(t)$ and $y(t)$ denote the time-varying levels of A's attraction to B and B's attraction to A respectively. The "interaction coefficients" in the rate DEs for $x(t)$ and $y(t)$ are unchanged, since their emotional patterns are still the same; the effect of the Wizard's intervention on the rates of change of their feelings for each other is then to add the "external" functions $f_1(t)$ and $f_2(t)$ respectively to these DEs, so that we get

$$x' = x - y + f_1(t) \qquad y' = 2x - y + f_2(t).$$

To ease your computational load, we'll tell you that the corresponding homogeneous system has solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ \cos t + \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t + \sin t \end{bmatrix}.$$

Whatever the Wizard intended, the effect on A and B of their scheduled ingestion of the potion turns out to be $f_1(t) = 3$ and $f_2(t) = -3$.

So that *apparently* Armand has a good reaction and Babette a bad reaction to it.

Guess a constant solution to find a particular solution to this inhomogeneous system.

What is the effect of the potion on A and B's situation? Is it positive or negative?

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