ES.1803 Problem Set 8, Spring 2024 Solutions

Part II   110 + 5 EC points

Problem 1 (Topic 24) (Armand and Babette are reprogrammed) (30: 10,5,10,5)
Having given up hope that the Therapist or Wizard could help them, Armand and Babette turned to a Pharmacist. She gave them a drug (patent pending) that completely reprogrammed their emotions.

(a) *Unfortunately the drug's effects are permanent and set their system to*

$$x' = -2x + y \qquad y' = x - 2y$$

Solve this system and describe what will happen to their love if nothing more is done.

Solution: In matrix form the system is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

The characteristic equation is $\lambda^2 + 4\lambda + 3 = 0$. This has roots $\lambda = -1, -3$ with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since both eigenvalues are negative, their love appears doomed to dwindle to nothing.

(b) *The Pharmacist gave them another drug in a time release capsule. This drug boosts the attraction they each feel for each other. Unfortunately Armand did not tolerate the drug – their armour dried out and they became listless. So only Babette could take it.*

Being DE Armadillos, one of their days is 2π units of time. Babette takes the drug once each day. Between the time release and her body's metabolism, the amount in her bloodstream varies and the boost in attraction follows a period 2π triangle wave

$$\text{tri}(t) = \begin{cases} -t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

Thus the equations of their attraction are: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \text{tri}(t) \end{bmatrix}$.

Use the top equation to eliminate y from the bottom equation and get equations

$$x'' + 4x' + 3x = \text{tri}(t), \qquad y = x' + 2x.$$

Solution: Written out, the two equations are

$$\begin{aligned} x' &= -2x + y \\ y' &= x - 2y + \text{tri}(t) \end{aligned}$$

The top equation can be solved for y: $y = x' + 2x$. We then substitute this in the bottom equation to eliminate y:

$$y' = x'' + 2x' = x - 2(x' + 2x) + \text{tri}(t).$$

A little bit of algebra gives: $x'' + 4x' + 3x = \text{tri}(t)$. This and the equation $y = x' + 2x$ are the two equations we were asked to derive.

(c) *Find the general solution to these equations for x and y.*

Solution: Using the Fourier series for $\text{tri}(t)$ the equation for x can be written as

$$x'' + 4x' + 3x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}.$$

The characteristic roots are -1 and -3, so the homogeneous solution is $x_h(t) = c_1 e^{-t} + c_2 e^{-3t}$.

We find a particular solution by solving with each term of the Fourier series and using linearity. With this in mind, solve

$$x_n'' + 4x_n' + 3x_n = \cos(nt).$$

The sinusoidal response formula gives a solution: $x_{n,p}(t) = \dfrac{\cos(nt - \phi(n))}{|P(in)|}$, where

$|P(in)| = \sqrt{(3-n^2)^2 + 16n^2}$ and $\boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}(4n/(3-n^2)) \text{ in Q1 or Q2.}}$

We also solve $x_0'' + 4x_0' + 3x_0 = \pi/2$ to get $x_{0,p}(t) = \pi/6$.

Now using linearity we get a particular solution for x:

$$x_p(t) = \frac{\pi}{6} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{x_{n,p}}{n^2} = \frac{\pi}{6} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt - \phi(n))}{n^2\sqrt{(3-n^2)^2 + 16n^2}}$$

The general solutions are

$$x = x_h + x_p = c_1 e^{-t} + c_2 e^{-3t} + \frac{\pi}{6} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt - \phi(n))}{n^2\sqrt{(3-n^2)^2 + 16n^2}}$$

$$y = x' + 2x = c_1 e^{-t} - c_2 e^{-3t} + \frac{\pi}{3} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{2\cos(nt - \phi(n)) - n\sin(nt - \phi(n))}{n^2\sqrt{(3-n^2)^2 + 16n^2}}$$

(d) *Give a sketch that approximates the graphs of the periodic solutions for x and y. Be sure to explain how you got the approximation. Finally, say if the drug has a beneficial effect.*

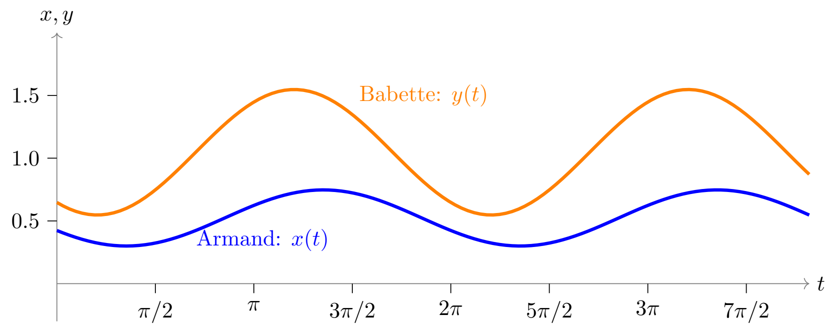
Solution: For the periodic solution, we ignore the transient terms. The Fourier coefficients decay fast: like $1/n^4$ for x, and like $1/n^3$ for y . Writing out the first few terms of each expansion we get:

$$x_p(t) = \frac{\pi}{6} - \frac{4}{\pi}\Big(0.224\cos(t - \phi(1)) + 0.008\cos(3t - \phi(3)) + 0.001\cos(5t - \phi(5)) + \ldots\Big)$$

$$y_p(t) = \frac{\pi}{3} - \frac{4}{\pi}\Big(0.447\cos(t - \phi(1)) - 0.224\sin(t - \phi(1)) + 0.017\cos(3t - \phi(3)) - 0.025\sin(3t - \phi(3)) + 0.003\cos(5t - \phi(5)) - 0.007\sin(5t - \phi(5)) + \ldots\Big)$$

For sketching approximations of the graphs, we can ignore the $n \geq 3$ terms. We plotted:

$$x(t) \approx \frac{\pi}{6} - \frac{4}{\pi}\Big(0.224\cos(t - \phi(1))\Big)$$
$$y(t) \approx \frac{\pi}{3} - \frac{4}{\pi}\Big(0.447\cos(t - \phi(1)) - 0.224\sin(t - \phi(1))\Big)$$



The drug causes the attraction of each one to the other to stay in the positive range. Assuming they are truly meant for each other, this is beneficial!

Problem 2 (15 points)
Read the Topic 25 notes. You'll be glad you did. The answer to this problem should be 'Yes, I read the notes. One thing I didn't understand was: (fill in the blank).*'*

Solution: Very informative!

Problem 3 (Topic 25) (20: 5,5,5,5) **Linearity and homogeneity**
We've used superposition when solving the wave and heat equations. In this problem you'll prove that the equations and boundary conditions are linear. Each answer should take only 1 or 2 lines. The important point here is to understand why we care about this.

Consider the heat equation

$$u_t = c\, u_{xx}, \quad \textit{where } c > 0 \textit{ is a constant.} \qquad \text{(H)}$$

Also consider the homogeneous boundary conditions

$$u(0,t) = 0 \ \ \textit{and} \ \ u_x(L,t) = 0. \qquad \text{(HBC)}$$

Finally, let $\mathcal{T}$ be the partial differential operator defined by

$$\mathcal{T}u = u_t - cu_{xx}.$$

(a) *Show the heat equation (H) can be written $\mathcal{T}u = 0$.*

Solution: This is trivial.

(b) *Show $\mathcal{T}$ is a linear operator.*

Solution: Linearity means that for functions $u(x,t)$ and $v(x,t)$ and constants c_1 and c_2 we have $\mathcal{T}(c_1u + c_2v) = c_1\mathcal{T}u + c_2\mathcal{T}v$. This is easy to verify:

$$\mathcal{T}(c_1u+c_2v) = c_1u_t + c_2v_t - c_1cu_{xx} - c_2cv_{xx} = c_1(u_t - cu_{xx}) + c_2(v_t - cv_{xx}) = c_1\mathcal{T}u + c_2\mathcal{T}v.$$

(c) *Use Part (b) to show that if u_1 and u_2 are solutions to (H) then so is $u_1 + u_2$.*

Solution: By (b) $\mathcal{T}(u_1 + u_2) = \mathcal{T}(u_1) + \mathcal{T}(u_2) = 0 + 0$. QED

(d) *Show that two solutions to the homogeneous boundary conditions (HBC) can be superpositioned to give another solution to (HBC).*

Solution: If $u(x,t)$ and $v(x,t)$ both satisfy (HBC) it is clear that so does $u + v$.

Problem 4 (Topic 25) (30 + 5EC: 5,10,10,5,5EC)

The Heat Equation and Solar Energy Storage

The example we have in mind is a solar pond, which can store heat to be used to generate electricity. Without salt, the hot water rises and the cold water sinks (called convection), causing much of the heat to be lost through the top of the pond. Adding salt gives the water a salinity gradient which damps down the convection because the hotter bottom water is also heavier, so it doesn't tend to rise. In this case, the movement of heat is mostly by conduction and results in much less heat loss.

Let $u(x,t)$ be the temperature of the pond at depth x at time t. If no heat is being added to the pond, the temperature is modeled by the PDE (H) in Problem 3.

If the sun is heating the pond, then we need to add input to the model. We'll assume that it adds heat to the water linearly with respect to depth. This is modeled by the inhomogeneous, PDE (I), with inhomogeneous boundary conditions (IBC):

$$u_t = cu_{xx} + a(L - x) \tag{I}$$

$$u(0,t) = T_0 \text{ and } u_x(L,t) = 0. \tag{IBC}$$

Here, L is the depth of the pond, $a > 0$ is a constant which determines the rate of heating and T_0 is the temperature of the air.

Physically the first boundary condition says the temperature of the water surface is the same as that of the air and the second one says the earth acts as an insulator, so no heat is transfered from the bottom of the pond into the earth.

(a) (Superposition principles.) *(i) Suppose $u_h(x,t)$ is a solution to (H) and $u_p(x,t)$ is a solution to (I). Show $u = u_p + u_h$ is also a solution to (I).*

(ii) Show that if $u_h(x,t)$ satisfies (HBC) and $u_p(x,t)$ satisfies (IBC) then $u = u_p + u_h$ also satisfies (IBC).

Here (H) and (HBC) refer to the equations in Problem 3.

Solution: (i) This is just the linearity of $\mathcal{T}$.

$$\mathcal{T}(u_p + u_h) = \mathcal{T}(u_p) + \mathcal{T}(u_h) = a(L - x) + 0. \qquad QED$$

(ii) We have to show the boundary conditions can be superpositioned. This is easy, $u_p(0,t) = T_0$ and $u_h(0,t) = 0 \Rightarrow u(0,t) = u_p(0,t) + u_h(0,t) = T_0$.

Likewise $u_x(\pi,t) = (u_p)_x(\pi,t) + (u_h)_x(\pi,t) = 0$. QED

For Parts b-e let $c = 1$, $a = 1$ and $L = \pi$.

(b) *Find the steady-state solution of the PDE (I), which satisfies (IBC). This will be the temperature profile of the pond after a sufficiently long time has elapsed. Hint: At the steady state, the solution does not depend on time.*

Also, show that at this steady state, the water in the pond is hottest at the bottom.

Solution: The steady state solution does not depend on time, so we can write $u(x,t) = X(x)$. Thus, the PDE (I) becomes

$$0 = X''(x) + (\pi - x).$$

This is an 18.01 problem: $X(x) = \dfrac{(x-\pi)^3}{6} + c_1 x + c_2$.

The boundary conditions (IBC) are $X(0) = T_0$, $X'(\pi) = 0$. So,

$$\begin{aligned} X(0) &= -\frac{\pi^3}{6} + c_2 = T_0 \Rightarrow c_2 = T_0 + \frac{\pi^3}{6} \\ X'(\pi) &= c_1 = 0 \end{aligned}$$

So the steady state solution is

$$u_p(x,t) = X(x) = \frac{(x-\pi)^3}{6} + T_0 + \frac{\pi^3}{6}.$$

The hottest pond temperature is the maximum of $X(x)$ on $[0,\pi]$. It's easy to see that $X(x)$ is maximized on $[0,\pi]$ at $x = \pi$. That is, the maximum temperature is at the bottom of the pond.

(c) *Find the general solution $u(x,t)$ to (H) + (HBC) by using the Fourier separation-of-variables method.*

Solution: We are solving:

$$u_t(x,t) = u_{xx}(x,t) \qquad \text{(H)}$$
$$u(0,t) = 0,\ u_x(\pi,t) = 0. \qquad \text{(HBC)}$$

Step 1. Find separated solutions to Equation (H)

Try $u(x,t) = X(x)T(t)$.

Plug this into (H) to get: $XT' = X''T \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T} = \text{ constant } = -\lambda$.

(As always, if a function of x = function of t then they both must be constant functions. We call the constant $-\lambda$.) So, we have two ODEs:

$$X'' + \lambda X = 0, \quad T' + \lambda T = 0.$$

Break into cases.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$, $T(t) = ce^{-\lambda t}$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c$.

Case (iii) $\lambda < 0$: Can skip this case, it never produces nontrivial modal solutions.

Step 2. Find modal solutions (separated solutions, which also satisfy (HBC)).

For separated solutions, the homogeneous boundary conditions (HBC) are $X(0) = 0$, $X'(\pi) = 0$. Again, we examine the cases.

Case (i): $X(0) = a = 0, \quad X'(\pi) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0.$

Since $a = 0$, the second condition becomes $b\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$.

If $b = 0$, then $X(x) = 0$ and we have the trivial solution.

If $\cos(\sqrt{\lambda}\,\pi) = 0$, then $\sqrt{\lambda} = n/2$, with n an odd integer.

Thus we have modal solutions to $(H) + (HBC)$

$$u_n(x,t) = b_n e^{-\frac{n^2}{4}t}\sin\left(\frac{nx}{2}\right), \qquad \text{where } n \text{ is odd.}$$

Case (ii): $X(0) = a = 0, \quad X'(\pi) = b = 0.$

So, this case provides no nontrivial modal solutions.

Case (iii): Ignored –no nontrivial solutions.

Step 3. Write down the general homogeneous solution.

Using superposition, we combine all the modal solutions to get the general solution to (H), which also satisfies (HBC):

$$\boxed{u_h(x,t) = \sum_{n\,odd} u_n(x,t) = \sum_{n\,odd} b_n \sin\left(\frac{nx}{2}\right)\cdot e^{-\frac{n^2}{4}t}}.$$

(d) *What is the general solution to the inhomogeneous system $(I) + (IBC)$?*

Hint: The steady-state solution is a particular solution of the original PDE.

Solution: By the superposition principle in Part (a), the general inhomogeneous solution is

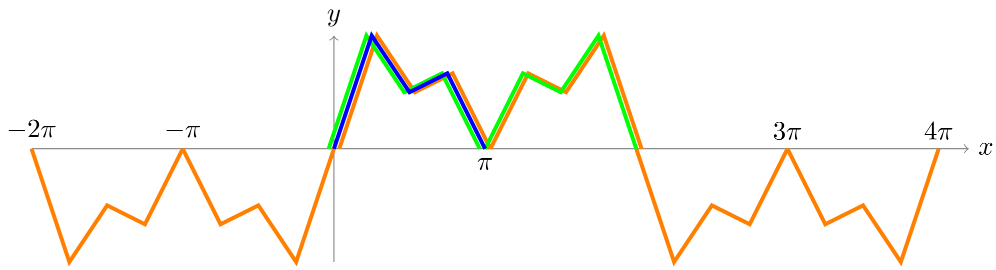
$$\boxed{u(x,t) = u_p(x,t) + u_h(x,t) = \frac{(x-\pi)^3}{6} + T_0 + \frac{\pi^3}{6} + \sum_{n\,odd} b_n \sin\left(\frac{nx}{2}\right)\cdot e^{-\frac{n^2}{4}t}.}$$

(e) (Extra credit (Hard!)) *Your answer in Part (c) should involve a sine series of the form $\sum_{n\ odd} b_n \sin(nx/2)$. An initial condition $v(x,0) = f(x)$ could be used to find the coefficients. This is not, in general, the Fourier series of the odd period 2π extension of $f(x)$, but rather the Fourier series of a different extension of $f(x)$. Describe that extension.*

Solution: The base period of the series $\sum_{n \text{ odd}} b_n \sin(nx/2)$ is 4π. This motivates the following extension: First extend $f(x)$ from the interval $[0,\pi]$ to $[0,2\pi]$ by mirroring the graph of $f(x)$ over $[0,\pi]$ onto $[\pi, 2\pi]$ (see figure), call it $f_2(x)$:

$$f_2(x) = \begin{cases} f(x) & \text{for } 0 < x < \pi \\ f(x-\pi) & \text{for } \pi < x < 2\pi \end{cases}$$

The coefficients b_n are the Fourier sine coefficients of f_2, i.e., the Fourier coefficients of the odd period 4π extension of f_2 –call this $\tilde{f}_{2,o}$.

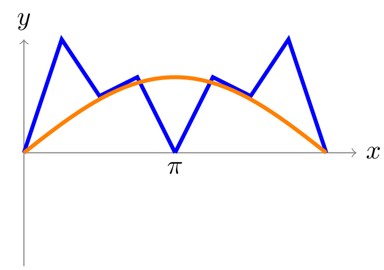


$f(x)$ on $[0, \pi]$, $f_2(x)$ on $[0, 2\pi]$, $\tilde{f}_{2,\mathrm{o}}$ full graph

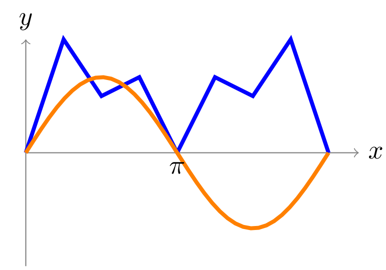
Since $\tilde{f}_{2,\mathrm{o}}(x)$ is odd, its Fourier series is of the form $\sum_{n=1}^{\infty} b_n \sin(nx/2)$. The coefficients are

$$b_n = \frac{2}{2\pi} \int_0^{2\pi} f_2(x) \sin(nx/2)\, dx.$$

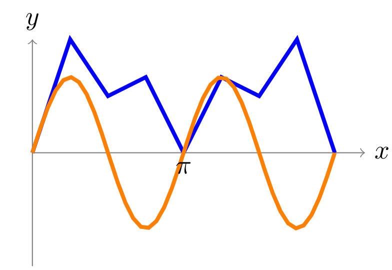
The symmetry in the figures below, illustrate why $b_n = 0$ when n is even.



$b_1 = \frac{1}{\pi} \int_0^{2\pi} f_2(x) \sin(x/2)\, dx \neq 0.$



$b_2 = \frac{1}{\pi} \int_0^{2\pi} f_2(x) \sin(x)\, dx = 0.$



$b_4 = \frac{1}{\pi} \int_0^{2\pi} f_2(x) \sin(x/2)\, dx = 0.$

Note also that, by symmetry, $b_1 = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x) \sin(x)\, dx$. Likewise, for the other odd coefficients.

Problem 5 (Topic 25) (15: 10,5)
Realistically strings don't vibrate forever. To model this we can add a damping term to the wave equation. If we clamp the ends, we get boundary conditions. Finally, we can add initial conditions. Altogether we get the following PDE with boundary and initial conditions.

$$y_{tt} + b\, y_t = a^2\, y_{xx} \quad \text{for } 0 \le x \le L,\ t > 0 \qquad \text{(PDE)}$$
$$y(0,t) = y(L,t) = 0 \qquad \text{(BC)}$$
$$y(x,0) = f(x),\ y_t(x,0) = 0. \qquad \text{(IC)}$$

For this problem take $L = 1,\quad a = 2,\quad b = 7$ *and leave* $f(x)$ *arbitrary.*

(a) *Use separation of variables to solve the PDE with boundary and initial conditions. (You will need to leave the Fourier sine or cosine series of* f *in terms of arbitrary coefficients. Also, be careful when finding the function* $T(t)$ *for small values of* n*.)*

Solution: Step 1. Try separated solutions to the PDE: $y(x,t) = X(x)T(t)$.

Substitute this into the PDE: $X(T'' + 7T') = 4X''T$.

Separating the variables gives $\dfrac{X''}{X} = \dfrac{T'' + 7T'}{4T} = -\lambda$.

(As always, if a function of x = function of t then they both must be constant functions. We call the constant $-\lambda$.) So, we have two ODEs:

$$X'' + \lambda X = 0, \quad T'' + 7T' + 4\lambda T = 0.$$

Break into cases.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. $T(t)$ will be found later.

Case (ii) $\lambda = 0$: $X(x) = a + bx$.

Case (iii) $\lambda < 0$: Can skip this case, it never produces nontrivial modal solutions.

Step 2. Find modal solutions (separated solutions, which also satisfy the BC).

For separated solutions, the boundary conditions are $X(0) = 0, \quad X(1) = 0$.

Case (i) $X(0) = a = 0, \quad X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda})$.

Since $a = 0$, the second condition becomes $b\sin(\sqrt{\lambda}) = 0$.

If $b = 0$, then $X(x) = 0$ and we have the trivial solution.

If $\sin(\sqrt{\lambda}) = 0$, then $\sqrt{\lambda} = n\pi$ for some integer n.

Conclusion: $\lambda = n^2\pi^2$, $n = 1, 2, 3, ...,$ Thus, for each n we have a modal solution: $y_n = b_n \sin(n\pi x)\, T_n(t)$.

Next we solve for $T_n(t)$: In this case $\lambda = (n\pi)^2$

Characteristic equation: $r^2 + 7r + 4\lambda = 0$. So the roots are

$$r = -3.5 \pm \sqrt{(3.5)^2 - 4n^2\pi^2}.$$

It's important to check if the expression under the square root is positive or negative. In this case, because the damping constant is small, it is negative for all $n \geq 1$. We write the roots as

$$r = -3.5 \pm \beta_n\, i, \text{ where } \boxed{\beta_n = \sqrt{|3.5^2 - 4n^2\pi^2|}}.$$

This gives: $T_n(t) = e^{-3.5t}(c_n \cos(\beta_n t) + d_n \sin(\beta_n t))$ and we have the modal solutions

$$y_n(x,t) = \sin(n\pi x)e^{-3.5t}(c_n \cos(\beta_n t) + d_n \sin(\beta_n t)).$$

Case (ii) $X(0) = a = 0, \quad X(1) = a + b = 0$.

This implies $a = 0$, $b = 0$, i.e., this case only produces the trivial solution.

Case (iii) Can ignore.

Step 3. Use superposition to give the general solution to the PDE and BC.

$$\boxed{y(x,t) = \sum y_n(x,t) = \sum \sin(n\pi x)e^{-3.5t}(c_n \cos(\beta_n t) + d_n \sin(\beta_n t))}.$$

Step 4. Use the initial conditions to find values for the coefficients.

$$y(x,0) = \sum \sin(n\pi x)c_n = f(x).$$

So the coefficients c_n are the Fourier sine coefficients of $f(x)$:

$$\boxed{c_n = 2\int_0^1 \sin(n\pi x) f(x)\, dx}.$$

Next consider the IC: $y_t(x,0) = 0$:

$$y_t(x,0) = \sum \sin(n\pi x)(-3.5c_n + \beta_n d_n) = 0.$$

So, $(-3.5c_n + \beta_n d_n) = 0$, which implies $\boxed{d_n = 3.5c_n/\beta_n}$.

Combining the formulas for $y(x,t)$, c_n, d_n we have the solution to the problem.

(b) *What is the physical effect of the damping term? How is this seen in your solution?*

Solution: The physical effect of the damping term is to dissipate energy, causing the vibrations to die out. This is seen in the $e^{-3.5t}$ term in the solution.

End of problem set 8 solutions.