ES.1803 Problem Set 8, Spring 2024

Part I (16 points)

Topic 24 (M, Apr. 22) Linear ODEs with periodic input.
Read: Topic 24 notes.
Hand in: Part I problems 24.1ac, 24.2, 24.3ab (posted with psets).

Topic 25 (T, Apr. 23) PDEs; separation of variables.
Read: Topic 25 notes.
Hand in: Part I problems 25.1, 25.2, 25.3 (posted with psets).

Part II 110 + 5 EC points

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problem 1 (Topic 24) (Armand and Babette are reprogrammed) (30: 10,5,10,5)
Having given up hope that the Therapist or Wizard could help them, Armand and Babette turned to a Pharmacist. She gave them a drug (patent pending) that completely reprogrammed their emotions.

(a) Unfortunately the drug's effects are permanent and set their system to

$$x' = -2x + y \qquad y' = x - 2y$$

Solve this system and describe what will happen to their love if nothing more is done.

(b) The Pharmacist gave them another drug in a time release capsule. This drug boosts the attraction they each feel for each other. Unfortunately Armand did not tolerate the drug – their armour dried out and they became listless. So only Babette could take it.

Being DE Armadillos, one of their days is 2π units of time. Babette takes the drug once each day. Between the time release and her body's metabolism, the amount in her bloodstream varies and the boost in attraction follows a period 2π triangle wave

$$\text{tri}(t) = \begin{cases} -t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

Thus the equations of their attraction are: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \text{tri}(t) \end{bmatrix}$.

Use the top equation to eliminate y from the bottom equation and get equations

$$x'' + 4x' + 3x = \text{tri}(t), \qquad y = x' + 2x.$$

(c) Find the general solution to these equations for x and y.

(d) Give a sketch that approximates the graphs of the periodic solutions for x and y. Be sure to explain how you got the approximation. Finally, say if the drug has a beneficial effect.

Problem 2 (15 points)
Read the Topic 25 notes. You'll be glad you did. The answer to this problem should be 'Yes, I read the notes. One thing I didn't understand was: *(fill in the blank)*.'

Problem 3 (Topic 25) (20: 5,5,5,5) **Linearity and homogeneity**
We've used superposition when solving the wave and heat equations. In this problem you'll prove that the equations and boundary conditions are linear. Each answer should take only 1 or 2 lines. The important point here is to understand why we care about this.

Consider the heat equation

$$u_t = c\,u_{xx}, \text{ where } c > 0 \text{ is a constant.} \tag{H}$$

Also consider the homogeneous boundary conditions

$$u(0,t) = 0 \text{ and } u_x(L,t) = 0. \tag{HBC}$$

Finally, let $\mathcal{T}$ be the partial differential operator defined by

$$\mathcal{T}u = u_t - cu_{xx}.$$

(a) Show the heat equation (H) can be written $\mathcal{T}u = 0$.

(b) Show $\mathcal{T}$ is a linear operator.

(c) Use Part (b) to show that if u_1 and u_2 are solutions to (H) then so is $u_1 + u_2$.

(d) Show that two solutions to the homogeneous boundary conditions (HBC) can be superpositioned to give another solution to (HBC).

Problem 4 (Topic 25) (30 + 5EC: 5,10,10,5,5EC)
The Heat Equation and Solar Energy Storage
The example we have in mind is a solar pond, which can store heat to be used to generate electricity. Without salt, the hot water rises and the cold water sinks (called convection), causing much of the heat to be lost through the top of the pond. Adding salt gives the water a salinity gradient which damps down the convection because the hotter bottom water is also heavier, so it doesn't tend to rise. In this case, the movement of heat is mostly by conduction and results in much less heat loss.

Let $u(x,t)$ be the temperature of the pond at depth x at time t. If no heat is being added to the pond, the temperature is modeled by the PDE (H) in Problem 3.

If the sun is heating the pond, then we need to add input to the model. We'll assume that it adds heat to the water linearly with respect to depth. This is modeled by the inhomogeneous, PDE (I), with inhomogeneous boundary conditions (IBC):

$$u_t = cu_{xx} + a(L - x) \tag{I}$$

$$u(0,t) = T_0 \text{ and } u_x(L,t) = 0. \tag{IBC}$$

Here, L is the depth of the pond, $a > 0$ is a constant which determines the rate of heating and T_0 is the temperature of the air.

Physically the first boundary condition says the temperature of the water surface is the same as that of the air and the second one says the earth acts as an insulator, so no heat is transfered from the bottom of the pond into the earth.

(a) (Superposition principles.) (i) Suppose $u_h(x,t)$ is a solution to (H) and $u_p(x,t)$ is a solution to (I). Show $u = u_p + u_h$ is also a solution to (I).

(ii) Show that if $u_h(x,t)$ satisfies (HBC) and $u_p(x,t)$ satisfies (IBC) then $u = u_p + u_h$ also satisfies (IBC).

Here (H) and (HBC) refer to the equations in Problem 3.

For Parts b-e let $c = 1$, $a = 1$ and $L = \pi$.

(b) Find the steady-state solution of the PDE (I), which satisfies (IBC). This will be the temperature profile of the pond after a sufficiently long time has elapsed. Hint: At the steady state, the solution does not depend on time.

Also, show that at this steady state, the water in the pond is hottest at the bottom.

(c) Find the general solution $u(x,t)$ to $(H) + (HBC)$ by using the Fourier separation-of-variables method.

(d) What is the general solution to the inhomogeneous system $(I) + (IBC)$?

Hint: The steady-state solution is a particular solution of the original PDE.

(e) (Extra credit (Hard!)) Your answer in Part (c) should involve a sine series of the form $\sum_{n \text{ odd}} b_n \sin(nx/2)$. An initial condition $v(x,0) = f(x)$ could be used to find the coefficients. This is not, in general, the Fourier series of the odd period 2π extension of $f(x)$, but rather the Fourier series of a different extension of $f(x)$. Describe that extension.

Problem 5 (Topic 25) (15: 10,5)
Realistically strings don't vibrate forever. To model this we can add a damping term to the wave equation. If we clamp the ends, we get boundary conditions. Finally, we can add initial conditions. Altogether we get the following PDE with boundary and initial conditions.

$$y_{tt} + b\,y_t = a^2\,y_{xx} \quad \text{for } 0 \le x \le L,\ t > 0 \qquad \text{(PDE)}$$
$$y(0,t) = y(L,t) = 0 \qquad \text{(BC)}$$
$$y(x,0) = f(x),\ y_t(x,0) = 0. \qquad \text{(IC)}$$

For this problem take $L = 1$, $a = 2$, $b = 7$ and leave $f(x)$ arbitrary.

(a) Use separation of variables to solve the PDE with boundary and initial conditions. (You will need to leave the Fourier sine or cosine series of f in terms of arbitrary coefficients. Also, be careful when finding the function $T(t)$ for small values of n.)

(b) What is the physical effect of the damping term? How is this seen in your solution?

End of problem set 8.