ES.1803 Problem Set 9, Spring 2024 Solutions

Part II 55 points

Problem 1 (Topic 27) (20: 10,10)
Recall our star-crossed armadillos Armand and Babette. Their time varying level of attraction was measured by $x(t)$ = Armand's attraction to Babette and $y(t)$ = Babette's attraction to Armand.

(a) *Suppose the equations modeling x and y are*

$$\begin{aligned} x' &= x - y \\ y' &= 2x - y \end{aligned}$$

(i) Find the general solution to these equations.

(ii) Sketch a phase portrait of the solutions.

(iii) Describe their relationship over time.

Solution: (i) The coefficient matrix is $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

Characteristic equation: $\det(A - \lambda I) = \lambda^2 + 1 = 0$.

Eigenvalues: $\lambda = \pm i$.

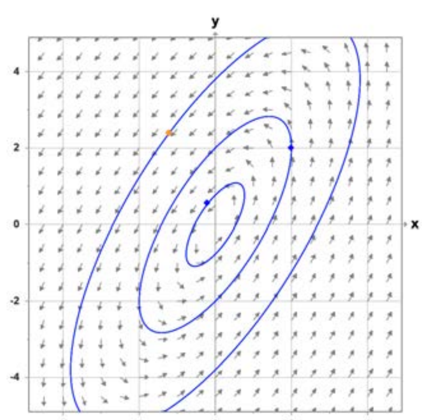
Baisc eigenvector for $\lambda = i$: $A - \lambda I = \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}$. We can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

Complex solution: $\mathbf{z}(t) = e^{it}\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} \cos(t) + i\sin(t) \\ (\cos(t)+\sin(t)) + i(-\cos(t)+\sin(t)) \end{bmatrix}$,

General solution (use the real *and* imaginary parts of $\mathbf{z}(t)$):

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos(t) \\ \cos(t)+\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ -\cos(t)+\sin(t) \end{bmatrix}$$

(ii) Pure imaginary eigenvalues tell us that the critical point at the origin is a center. The positive 2 in the lower left corner of the coefficient matrix tells us that the trajectories are in the counterclockwise direction. We used a version of the MIT Mathlet (`https://mathlets.org/mathlets/vector-fields/`) to make the phase portrait. A hand-drawn sketch is also a fine answer.



The system $x' = x - y, \quad y' = 2x - y.$

(iii) The phase portrait shows that their relationship moves in a never-ending cycle: A and B in love, B loves A but A hates B, they hate each other, A loves B but B hates A, back in love,...

(b) *Now suppose that their relationship is modeled by*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Sketch a phase portrait and describe their relationship over time.

Solution: The coefficient matrix is $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \lambda^2 + 3\lambda + 2 = 0$.

Eigenvalues: $\lambda = -1, -2$.

Basic eigenvectors:

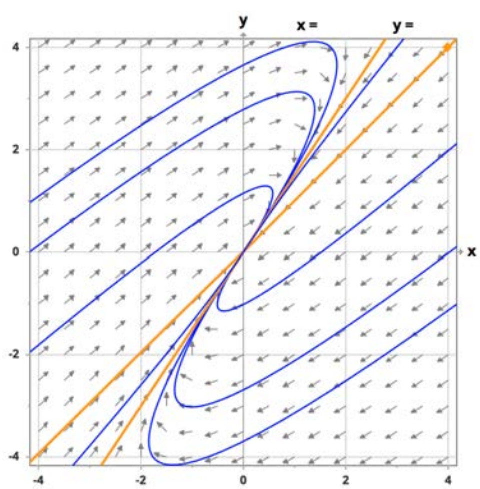
$\lambda = -1$: $\quad A - \lambda I = \begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$\lambda = -2$: $\quad A - \lambda I = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

General solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The negative real eigenvalues tell us that the critical point at the origin is a nodal sink. So all trajectories head to 0. Near the origin the trajectories are asymptotic to the eigenvector for $\lambda = -1$ and far from the origin they are asymptotically parallel to the eigenvector for $\lambda = -2$.



The system $x' = -4x + 2y, \quad y' = -3x + y.$

We see that their relationship is doomed to dwindle to nothing. Perhaps it's been years since they've even thought of one another.

Problem 2 (Topic 28) (15: 5,10) Armand and Babette go Non-Linear.

In a previous pset, The Wizard's potion did get them into the first quadrant (where happiness – or at least positive mutual attraction – resides), but it was only available as a short-term interim solution (for one thing, it has side-effects, like dry skin). However, DE armadillos, while at times introspective to the point of brooding, are also open to change. So they returned to the Therapist and after some process their coefficients have changed and their system has become nonlinear. Armand and Babette find that their feelings now vary according to rate equations

$$\begin{aligned} x' &= x - 2y + ex^2 \\ y' &= 5x - y + fy^2 \end{aligned}$$

with e, f certain constants. (DE armadillos are, of course, quite comfortable with parameters.) The addition of the square term in each rate equation is to account for an introspective affect, which always acts in the same direction, independent of the sign of the current state of attraction. This is small when the level is small and large when the level is large. (Hey, they're armadillos, who knows what makes them tick.)

(a) *Suppose $e = 1/4$ and $f = -1$. Find the critical points.*

Note: you will get a quartic polynomial. To help you solve it, we'll tell you that one root is 0 and another is a positive integer no larger than 5. There are only two critical points, but you'll need to find the other two roots of this polynomial and show they don't give critical points.

Solution: By definition, critical points are when $x' = 0$ and $y' = 0$:

Using $y' = 5x - y - y^2 = 0$, we get $x = (y + y^2)/5$. Substituting this into x', we get

$$x' = x - 2y + x^2/4 = (y + y^2)/5 - 2y + (y + y^2)^2/100 = 0 \;\Rightarrow\; y^4 + 2y^3 + 21y^2 - 180y = 0.$$

By inspection one root is $y = 0$ and, using the hint, we find another is $y = 4$.

Factoring: $x' = y(y - 4)(y^2 + 6y + 45) = 0 \quad \Rightarrow y = 0,\ 4,\ -3 \pm 6i.$

Only the real roots give critical points: $(0, 0)$ and $(4, 4)$.

(b) *Find the linearized system at each critical point. Then combine the linearized systems in one sketch giving your best guess for the phase portrait of the non-linear system.*

Show all work clearly.

Solution: We need the Jacobian for the system

$$\begin{aligned} x' &= x - 2y + x^2/4 &&= f(x, y) \\ y' &= 5x - y - y^2 &&= g(x, y) \end{aligned}$$

So, $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1 + x/2 & -2 \\ 5 & -1 - 2y \end{bmatrix}$.

Linearize at each critical point:

At $(0, 0)$: $\quad J(0, 0) = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}$.

Characteristic equation: $\quad \lambda^2 + 9 = 0$.

Eigenvalues: $\quad \lambda = \pm 3i$.

So the linearized system has a **center** at $(0,0)$. Since this is not structurally stable, the nonlinear system is either a center, spiral source or spiral sink.

The lower left entry of the matrix is positive, so trajecrories turn counterclockwise.

At $(4,4)$: $J(4,4) = \begin{bmatrix} 3 & -2 \\ 5 & -9 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 6\lambda - 17 = 0$

Eigenvalues: $\lambda = -3 \pm \sqrt{26}$.

So the linearized system has a **saddle** at $(4,4)$. Since this is structurally stable, the nonlinear system is also a saddle. This is an unstable critical point.

(See the plot in the next problem (rest assured my hand sketch was indistinguishable from it). Note, the critical point at (0,0) is not structurally stable, and the applet
`https://web.mit.edu/jorloff/www/OCW-ES1803/vectorFields.html`
is not precise enough to draw anything but a closed loop near the origin. So, I could only guess that it was a spiral sink.

Problem 3 (Topic 29) (5)
Classify as structurally stable or not stable each of the critical points found in Problem 2. Based on this what can you say for sure about the behavior of the trajectories near each critical point? How does this relate to your hand sketch of the phase portrait.

Solution: The linearized critical point at $(0,0)$ is a center, this is not structurally stable. A linearized center might be (in the nonlinear picture) a center or a spiral in or out.

The linearized critical point at $(4,4)$ is a saddle. This is structurally stable, so the nonlinearized system is also a saddle.

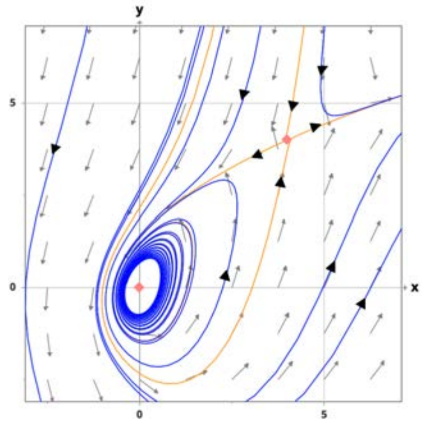
Near the structurally stable critical point our sketch is guaranteed to be qualitatively accurate. Near the structurally unstable point it is not, although our use of the applet gives us some hope.

Problem 4 (Topic 28) (15: 7,8)
(a) *Use the applet*
`https://web.mit.edu/jorloff/www/OCW-ES1803/vectorFields.html`
to obtain a phase portrait for the system in Problem 2. Experiment with enough initial values to get the full picture, and compare these results with those obtained above by hand.

Be sure to adjust the window as needed to get a good-looking set of trajectories.

Print your best picture and put arrows on some trajectories in the direction of increasing time.

Solution:



(b) *Use the results of Part (a) to analyze the behavior of the solutions to this system. First sketch in the two trajectories going to the critical point in the first quadrant that divide the quadrant into two regions; one captured by the critical point at 0 and the other ... (behaving as it does). Then, interpret the results. In particular, for the long run behavior, explain why A and B's friends will always find them exasperating.*

Solution: The two trajectories going asymptotically to $(4, 4)$ are sketched in the plot shown in Part (a). Trajectories that start 'inside' these two are attracted to the critical point at $(0, 0)$; it's hard to tell from the plot shown whether they all spiral in to $(0, 0)$ or spiral in to some limit cycle. (Playing with the applet leads me to guess that they do in fact spiral in to $(0, 0)$.) Trajectories that start 'outside' the two dividing trajectories fall into the influence of the saddle point at $(4, 4)$. That is, they head off along an asymptote with a positive slope.

Why is this exasperating? Well, won't these two ever settle down into being 'old-marrieds'? Their future holds one of two possibilities. One choice is their linear system cycle of love and hate. True, they look to be spiraling down to a complete neutrality at (0,0), but this will take so long that their friends will have long since stopped listening to their up-and-down melodramatic tales. The other choice is probably worse, they head off on a dizzying trajectory of ever-increasing infatuation. I mean, they'll become middle-aged armadillos ($t \to \infty$), they'll have teen-aged children. Won't these embarrassing public displays of affection ever stop?

End of problem set 9 solutions.