ES.1803 Problem Set 9, Spring 2024

(Not to be turned in)

Part I (48 points)

Topic 27 (M, Apr. 29) Qualitative behavior of linear systems.
Read: Topic 27 notes.
Hand in: Part I problems 27.1ab, 27.2abd, 27.3abcde (posted with psets).

Topic 28 (R, May 2) Qualitative behavior of non-linear systems; linearization.
Read: Topic 28 notes.
Hand in: Part I problems 28.1b, 28.2b, 28.3ab, 28.4a, 28.5, 28.6, 28.8, 28.9 (posted with psets).

Topic 29 (M, May 6) Structural stability.
Read: Topic 29 notes.)
Hand in: Part I problems 29.1b, 29.2a (posted with psets).

Topic 30 (T, May 7) Applications to population models: Volterra's principle.
Read: Topic 30 notes.
Hand in: Part I problems 30.2, 30.4, 30.5 (posted with psets).

Part II 55 points

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problem 1 (Topic 27) (20: 10,10)
Recall our star-crossed armadillos Armand and Babette. Their time varying level of attraction was measured by $x(t)$ = Armand's attraction to Babette and $y(t)$ = Babette's attraction to Armand.

(a) Suppose the equations modeling x and y are

$$x' = x - y$$
$$y' = 2x - y$$

(i) Find the general solution to these equations.

(ii) Sketch a phase portrait of the solutions.

(iii) Describe their relationship over time.

(b) Now suppose that their relationship is modeled by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Sketch a phase portrait and describe their relationship over time.

Problem 2 (Topic 28) (15: 5,10) Armand and Babette go Non-Linear.
In a previous pset, The Wizard's potion did get them into the first quadrant (where happiness – or at least positive mutual attraction – resides), but it was only available as a

short-term interim solution (for one thing, it has side-effects, like dry skin). However, DE armadillos, while at times introspective to the point of brooding, are also open to change. So they returned to the Therapist and after some process their coefficients have changed and their system has become nonlinear. Armand and Babette find that their feelings now vary according to rate equations

$$x' = x - 2y + ex^2$$
$$y' = 5x - y + fy^2$$

with e, f certain constants. (DE armadillos are, of course, quite comfortable with parameters.) The addition of the square term in each rate equation is to account for an introspective affect, which always acts in the same direction, independent of the sign of the current state of attraction. This is small when the level is small and large when the level is large. (Hey, they're armadillos, who knows what makes them tick.)

(a) Suppose $e = 1/4$ and $f = -1$. Find the critical points.

Note: you will get a quartic polynomial. To help you solve it, we'll tell you that one root is 0 and another is a positive integer no larger than 5. There are only two critical points, but you'll need to find the other two roots of this polynomial and show they don't give critical points.

(b) Find the linearized system at each critical point. Then combine the linearized systems in one sketch giving your best guess for the phase portrait of the non-linear system.

Show all work clearly.

Problem 3 (Topic 29) (5)
Classify as structurally stable or not stable each of the critical points found in Problem 2. Based on this what can you say for sure about the behavior of the trajectories near each critical point? How does this relate to your hand sketch of the phase portrait.

Problem 4 (Topic 28) (15: 7,8)
(a) Use the applet
https://web.mit.edu/jorloff/www/OCW-ES1803/vectorFields.html
to obtain a phase portrait for the system in Problem 2. Experiment with enough initial values to get the full picture, and compare these results with those obtained above by hand.

Be sure to adjust the window as needed to get a good-looking set of trajectories.

Print your best picture and put arrows on some trajectories in the direction of increasing time.

(b) Use the results of Part (a) to analyze the behavior of the solutions to this system. First sketch in the two trajectories going to the critical point in the first quadrant that divide the quadrant into two regions; one captured by the critical point at 0 and the other ... (behaving as it does). Then, interpret the results. In particular, for the long run behavior, explain why A and B's friends will always find them exasperating.

End of problem set 9.