ES.1803 Quiz 1 Solutions, Spring 2024

Problem 1. (15 points)

Find the general solution to $\frac{dx}{dt} + tx = t$.

Solution: This is a first-order linear DE. The associated homogeneous solution is

$$x_h(t) = e^{-\int p(t)\,dt} = e^{-\int t\,dt} = e^{-t^2/2}.$$

Therefore, the variation of parameters formula gives the general solution

$$x(t) = x_h(t)\left(\int \frac{q(t)}{x_h(t)}\,dt + C\right) = e^{-t^2/2}\left(\int te^{t^2/2}\,dt + C\right) = e^{-t^2/2}\left(e^{t^2/2} + C\right) = \boxed{1 + Ce^{-t^2/2}}.$$

Problem 2. (15 points)

(a) (10) *Find the solution to the initial value problem:* $\frac{dx}{dt} + 3x = 5, \quad x(2) = 7.$

Solution: This is a first-order, linear DE. The homogeneous solution is

$$x_h(t) = e^{-\int p(t)\,dt} = e^{-\int 3\,dt} = e^{-3t}.$$

Therefore, the variation of parameters formula gives the general solution

$$x(t) = x_h(t)\left(\int \frac{f(t)}{x_h(t)}\,dt + C\right) = e^{-3t}\left(\int 5e^{3t}dt + C\right) = \frac{5}{3} + Ce^{-3t}.$$

The initial condition gives $x(2) = 5/3 + Ce^{-6} = 7$, so $C = \frac{16}{3}\cdot e^6$.

Answer: $\boxed{x(t) = \frac{5}{3} + \frac{16e^6}{3}e^{-3t}}$ or $\boxed{x(t) = \frac{5}{3} + \frac{16}{3}e^{-3(t-2)}.}$

(b) (5) *Solve the following differential equation with initial condition.* **Hint:** *Your answer to Part (a) will help considerably.*

$$\frac{dx}{dt} + 3x = \begin{cases} 5 & \text{for } t < 2 \\ 0 & \text{for } 2 < t \end{cases}, \qquad x(2) = 7.$$

Solution: Case $t < 2$: This is Part (a): $x(t) = \frac{5}{3} + \frac{16}{3}e^{-3(t-2)}$.

Case $t > 2$: DE: $x' + 3x = 0$, IC: $x(2) = 7$.

This is an exponential decay equation, so $x(t) = Ce^{-3(t-2)}$.

The IC gives: $x(2) = C = 7 \quad \longrightarrow x(t) = 7e^{-3(t-2)}$.

Full solution: $\boxed{x(t) = \begin{cases} \frac{5}{3} + \frac{16}{3}e^{-3(t-2)} & \text{if } t < 2 \\ 7e^{-3(t-2)} & \text{if } t > 2. \end{cases}}$

Problem 3. (15 points)
(a) (10)*A rectangular tank has cross-sectional area 4 m^2. Water flows into the top of the tank at a constant rate b in $\frac{m^3}{\text{min}}$, and out of the bottom at a rate proportional to the height of water in the tank.*

*Give the DE which governs the height of water in the tank. Be sure to define and give units for any symbols you introduce. (**Hint:** volume = height × cross-sectional area.)*

Do not solve the DE.

Solution: We are given information in terms of volume. $\dfrac{dV}{dt} = b - ky$, where b, in meter^3/min, is the rate water flows in. Here:

y, in meters, is the height of the water in the tank

k, in meter^2/min, is the constant of proportionality of the outflow.

Since $V = 4y$, we have $\boxed{4\dfrac{dy}{dt} = b - ky \Leftrightarrow \dfrac{dy}{dt} + \dfrac{k}{4}y = \dfrac{b}{4}.}$

(b) (5) *Assuming that the tank in Part (a) has sufficient capacity, the height of the water in the tank will reach an equilibrium. What is the equilibrium height?*

Solution: At equilibrium the rate of change $y' = 0$. Putting $y' = 0$ into the DE we find $0 = b - ky$, so $\boxed{y = b/k.}$

Problem 4. (15 points)
Consider the family of curves $y = Cx^2$. Find the family of orthogonal trajectories.

Hint: *Isolate the C before differentiating.*

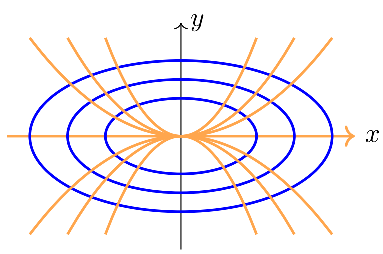
Solution: Following the hint: $yx^{-2} = C$.

Differentiating: $y'x^{-2} - 2yx^{-3} = 0 \quad \Rightarrow y' = 2y/x$.

So the orthogonal family has $y' = -\dfrac{x}{2y}$.

This is separable: $2y\,dy = -x\,dx \quad \Rightarrow y^2 = -x^2/2 + C$

The orthogonal family of trajectories is $\boxed{y^2 + \dfrac{x^2}{2} = C}$ (ellipses).



Problem 5. (10 points)
(a) (5) *True or false: Multiplying any solution of $\ddot{y} + 3t\dot{y} + 5y = t$ by 7 produces another solution to the same differential equation.*

You need to give a short explanation of your reasoning.

Solution: False: This is an inhomegeneous equation, i.e., $(7y)'' + 3t(7y)' + 5(7y) = 7t$, not t.

(b) (5) *The equation* $\ddot{y} + \left(4 - \frac{1}{t}\right)\dot{y} + \left(4 - \frac{2}{t} - \frac{3}{t^2}\right)y = f(t)$ *has two solutions:*

$$y_1(t) = e^{2t}t^3 + e^{-2t}t^3 \quad \textit{and} \quad y_2(t) = e^{2t}t^3 - e^{-2t}t^3$$

Give one nontrivial solution to the homogeneous equation

$$\ddot{y} + \left(4 - \frac{1}{t}\right)\dot{y} + \left(4 - \frac{2}{t} - \frac{3}{t^2}\right)y = 0.$$

Solution: Since the equation is linear, the superposition principle tells us that the difference of the two solutions is a homogeneous solution: $\boxed{y = 2e^{-2t}t^3}$ is a homogeneous solution. (Any multiple of this will also work.)