ES.1803 Problem Section Problems for Quiz 1, Spring 2024 Solutions

Topic 1: Introduction, modeling, separation of variables

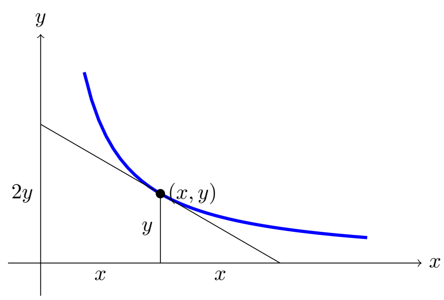
Problem 1.1. *(Here's the second geometry example in the notes for Topic 1.)*

$y = y(x)$ is a curve in the first quadrant. The part of the tangent line in the first quadrant is bisected by the point of tangency. Find and solve the DE for this curve.

Solution: From the picture: the slope of the tangent $= \dfrac{dy}{dx} = \dfrac{-y}{x}$.

Separate variables: $\dfrac{dy}{y} = -\dfrac{dx}{x}$.

Integrate: $\ln|y| = -\ln|x| + C \;\Rightarrow\; \boxed{y = C/x.}$



Problem 1.2. *Consider the family of all lines whose y-intercept is twice the slope.*

(a) *Find a DE which has this family as its solutions.*

Solution: The lines are $y = mx + 2m = m(x+2)$. The key here is to end up with a DE in x and y that doesn't explicitly use the slope m. (The slope will be determined by the choice of C in the solution.) We have two different ways of finding m, so

$$\frac{dy}{dx} = m = \frac{y}{x+2}.$$

(b) *Find the orthogonal trajectories to the curves in Part (a). That is, find a family of functions whose graphs intersect all the lines in Part (a) orthogonally.*

Solution: Curves intersect orthogonally if their slopes (at points of intersection) are negative reciprocals. Taking the DE in Part (a) we get

$$\frac{dy}{dx} = -\frac{x+2}{y}.$$

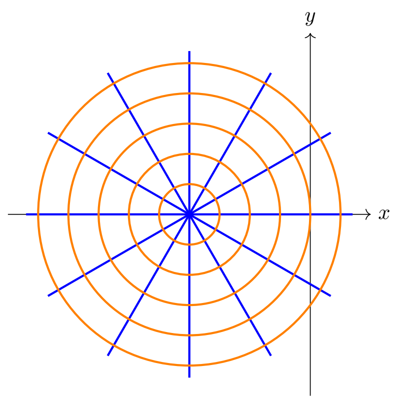
This is separable: $y\,dy = -(x+2)\,dx$.

Integrating: $y^2/2 = -(x+2)^2/2 + C$.

(Changing the meaning of C slightly.) We have $y^2 + (x+2)^2 = C$. This is a circle with center at (-2,0).

(c) *Sketch both families.*

Solution: Note that all the lines go through the point $(-2, 0)$, which is the center of the orthogonal circles.



Orthogonal lines and circles. The center is at $(-2, 0)$

Problem 1.3. *You deposit money in a bank at the rate of $1000/year. The money earns (continuous) 8% interest. Construct a DE to model the amount of money in the bank as a function of time; then solve the DE. Assume that at time 0 there is no money in the bank.*

Solution: Let $x(t)$ = amount in bank.

Quick answer: bank accounts have exponential growth, $1000/year is the input $\Rightarrow \dfrac{dx}{dt} = 0.08\,x + 1000$.

Slower answer: Over a small time Δt, the amount x, the interest rate and the deposit rate are all approximately constant. So, $\Delta x \approx 0.08\,x\,\Delta t + 1000\,\Delta t \;\Rightarrow\; \frac{\Delta x}{\Delta t} \approx 0.08x + 1000$
$\Rightarrow \dfrac{dx}{dt} = 0.08\,x + 1000$.

This is separable and first-order linear. You should solve this carefully, labeling each step. Use whichever technique you prefer.

The solution is $x(t) = 12500(e^{0.08t} - 1)$.

Problem 1.4. *Solve* $y' = f(x), \quad y(a) = y_0$.

Solution: $y(x) = \displaystyle\int_a^x f(u)\,du + y_0$.

Problem 1.5. *Solve* $y' = \sin(x^2), \quad y(0) = 1$. *Give the solution as a definite integral. (Note, you can't do the integral, but the solution is perfect for numerical computation by computer.)*

Solution: $y(x) = \displaystyle\int_0^x \sin(u^2)\,du + 1$.

Problem 1.6. *Solve* $\dfrac{du}{dt} = \sin t\,\cos^2 u, \quad u(0) = 0$.

Solution: Separate variables: $\sec^2 u\, du = \sin t\, dt$.

Integrate: $\tan u = -\cos t + C$.

IC: $0 = -1 + C \Rightarrow C = 1 \Rightarrow \boxed{\tan u = 1 - \cos t.}$

Problem 1.7. *(From Topic 1 notes.) Solve* $\dfrac{dy}{dx} = xy$.

Solution: Separating variables: $\dfrac{dy}{y} = x\, dx$. Therefore, $\displaystyle\int \frac{dy}{y} = \int x\, dx$, which implies $\ln|y| = \dfrac{x^2}{2} + C$.

As usual:

If $y > 0$, the solution is $y(x) = e^C e^{x^2/2}$.

If $y < 0$, the solution is $y(x) = -e^C e^{x^2/2}$.

The lost solution is $y(x) = 0$.

Putting all of these together: the general solution is $y(x) = Ke^{x^2/2}$.

Problem 1.8. *(From Topic 1 notes.) Solve* $\dfrac{dy}{dx} = x^3 y^2$.

Solution: Separating variables and integrating gives: $-\frac{1}{y} = \frac{x^4}{4} + C$. Solving for y we have

$$y = -\frac{4}{x^4 + 4C}.$$

There is also a lost solution: $y(x) = 0$.

Problem 1.9. *(From Topic 1 notes.) Solve* $y' + p(x)y = 0$.

Solution: We first rewrite this so that it's clearly separable: $\dfrac{dy}{y} = -p(x)\, dx$. After the usual separation and integration we have

$$\log(|y|) = -\int p(x)\, dx + C$$

Therefore, $|y(x)| = e^C e^{-\int p(x)\, dx}$ and $y(x) = 0$ is a lost solution. As usual, we can write the general solution as $y(x) = Ke^{-\int p(x)\, dx}$.

Topic 2: Linear systems, input-response

Problem 2.10. (Linear homogeneous)

(a) *Solve* $y' + ky = 0$.

Solution: This is our standard exponential decay equation: $y' = -ky$. So, $y(t) = Ce^{-kt}$.

(b) *Solve $y' + ty = 0$.*

Solution: This is separable: $\frac{dy}{y} = -t\,dt \quad \Rightarrow y(t) = Ce^{-t^2/2}$.

Problem 2.11. (Linear inhomogeneous)
(a) *Solve $y' + 2y = 2$.*

Solution: This is a first-order, linear, inhomogeneous DE. The associated homogeneous equation is $y_h' + 2y_h = 0$. This is our usual exponential decay equation. So, $y_h(t) = e^{-2t}$.
For the inhomogeneous DE, the input $q(t) = 2$. So the variation of parameters formula says

$$\begin{aligned} y(t) &= y_h(t) \int q(t)/y_h(t)\,dt + C\,y_h(t) \\ &= e^{-2t} \int \frac{2}{e^{-2t}} + Ce^{-2t} \\ &= e^{-2t} \int 2e^{2t}\,dt + Ce^{-2t} \\ &= e^{-2t}\left(e^{2t}\right) + Ce^{-2t} \\ &= \boxed{y = 1 + Ce^{-2t}} \end{aligned}$$

(Note: we could also have found the solution $y(t) = 1$ by inspection.)

(b) *Solve $y' + 2y = 2t$.*

Solution: Again, $y_h(t) = e^{-2t}$. Use variation of parameters:

$$y(t) = e^{-2t} \int 2te^{2t}\,dt + Ce^{-2t} = e^{-2t}\left(te^{2t} - \frac{e^{2t}}{2}\right) + Ce^{-2t} = \boxed{y = t - \frac{1}{2} + Ce^{-2t}}$$

(c) *Solve $y' + 2y = 5 + 2t$ using the earlier parts of this problem and superposition.*

Solution: The input to this DE is 2.5 times the input to Part (a) + the input to Part (b). So we can superpositon the answers to Parts (a) and (b) to get

$$y(t) = 2.5 + t - \frac{1}{2} + Ce^{-2t}.$$

Problem 2.12. (IVP using definite integrals)
Solve $xy' - e^x y = 0$, $y(1) = 2$ using definite integrals.

Solution: This is separable: $\frac{dy}{y} = \frac{e^x\,dx}{x}$. Because we can't compute $\int \frac{e^x}{x}\,dx$ in closed form, we need to give a definite integral solution.

$$\int_2^y \frac{du}{u} = \int_1^x \frac{e^v}{v}\,dv \quad \Rightarrow \log(y) - \log(2) = \int_1^x \frac{e^v}{v}\,dv.$$

Exponentiating we get: $\boxed{y(x) = 2e^{\int_1^x e^v/v\,dv}.}$

Problem 2.13. *Solve $y' + 2y = 2$; $y(1) = 1$.*

Solution: Variation of parameters gives $y(t) = 1 + Ce^{-2t}$. The initial condition then gives $y(1) = 1 = 1 + C(e^{-2})$. So, $C = 0$ and $y(t) = 1$.

Problem 2.14. *Show that $y' + y^2 = q$ does not satisfy the superposition principle.*

Solution: We'll do this with a specific counterexample: (It could just as easily be done generally.) Suppose $y_1' + y_1^2 = 1$ and $y_2' + y_2^2 = t$. If superposition were true, then we would have

$$(y_1 + y_2)' + (y_1 + y_2)^2 = 1 + t.$$

But

$$(y_1 + y_2)' + (y_1 + y_2)^2 = y_1' + y_1^2 + y_2' + y_2^2 + 2y_1y_2 = 1 + t + 2y_1y_2 \neq 1 + t.$$

So superposition doesn't hold.

Topic 3: Input response continued

Problem 3.15. *Solve the DE $x' + 2x = f(t)$, $x(0) = 0$, where* $f(t) = \begin{cases} 6 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } 1 \leq t < 2 \\ 6 & \text{for } 2 \leq t. \end{cases}$

Solution: First we solve the general cases (you can and should solve these by memory and inpsection).

IVP 1: $x' + 2x = 0$, $x(t_0) = b \quad \Rightarrow \; x(t) = be^{-2(t-t_0)}$.

IVP 2: $x' + 2x = 6$, $x(t_0) = b \Rightarrow x(t) = 3 + (b-3)e^{-2(t-t_0)}$.

For our problem:
Case $0 \leq t < 1$: DE: $x' + 2x = 6$, $x(0) = 0$.

So, using IVP 2, $x(t) = 3 - 3e^{-2t}$. For the next case: $x_1 = x(1) = 3(1 - e^{-2})$.

Case $1 \leq t < 2$: DE: $x' + 2x = 0$, $x(1) = x_1$.

So, using IVP 1, $x(t) = x_1 e^{-2(t-1)}$. For the next case: $x_2 = x(2) = x_1 e^{-2} = 3(e^{-2} - e^{-4})$.

Case $2 \leq t$: DE: $x' + 2x = 6$, $x(2) = x_2$.

So, using IVP 2, $x(t) = 3 + (x_2 - 3)e^{-2(t-2)}$.

Putting the cases together:

$$x(t) = \begin{cases} 3(1 - e^{-2t}) & \text{for } 0 \leq t < 1 \\ x_1 e^{-2(t-1)} \;=\; 3(1 - e^{-2})e^{-2(t-1)} & \text{for } 1 \leq t < 2 \\ 3 + (x_2 - 3)e^{-2(t-2)} \;=\; 3 + 3(-1 + e^{-2} - e^{-4})e^{-2(t-2)} & \text{for } 2 \leq t. \end{cases}$$
