ES.1803 Quiz 2 Solutions, Spring 2024

Problem 1. (50 points)

The differential operator in this problem is: $P(D) = 2D^2 + 2D + 5I$

Hint: The roots of $2r^2 + 2r + 5$ *are* $-\frac{1}{2} \pm \frac{3}{2}\, i$.

(a) (10) *Find the general* real-valued *solution to* $P(D)x = e^{-2t}$.

Solution: The DE is linear, 2^{nd} order, constant coefficient, with exponential input.

Particular solution Using the exponential response formula (ERF):

$$x_p(t) = \frac{e^{-2t}}{P(-2)} = \frac{e^{-2t}}{9}.$$

Homogeneous solution: $2x'' + 2x' + 5x = 0$.

We are given the characteristic roots: $r = -\frac{1}{2} \pm \frac{3}{2}\, i$.

Homogeneous solution: $\boxed{x_h(t) = c_1 e^{-t/2}\cos(3t/2) + c_2 e^{-t/2}\sin(3t/2).}$

General (real-valued) solution:

$$\boxed{x(t) = x_p(p) + x_h(t) = \frac{e^{-2t}}{9} + c_1 e^{-t/2}\cos(3t/2) + c_2 e^{-t/2}\sin(3t/2).}$$

(b) (10) *Find the general* real-valued *solution to* $P(D)x = 15t - 14$.

Solution: The DE is linear, 2^{nd} order, constant coefficient, with polynomial input.

Particular solution (using undetermined coefficients):

Trial solution: $x(t) = At + B$.

Substitution: $0 + 2A + 5(At + B) = 15t - 14 \quad \longrightarrow 5At + (2A + 5B) = 15t - 14$.

Equate coefficients: $5A = 15$ and $2A + 5B = -14 \quad \Rightarrow A = 3,\ B = -4$. So, $\boxed{x_p(t) = 3t - 4.}$

Homogeneous DE: Same as Part (a).

General (real-valued) solution:

$$\boxed{x(t) = x_p(t) + x_h(t) = (3t - 4) + c_1 e^{-t/2}\cos(3t/2) + c_2 e^{-t/2}\sin(3t/2).}$$

(c) (10) *Find the periodic solution to* $P(D)x = \omega\cos(\omega t)$.

Solution: Sinusoidal response formula (SRF): $x_p(t) = \dfrac{\omega\cos(\omega t - \phi(\omega))}{|P(i\omega)|}$.

Polar form: $P(i\omega) = 5 - 2\omega^2 + 2i\omega$:

$|P(i\omega)| = \sqrt{(5 - 2\omega^2)^2 + 4\omega^2}$ and $\boxed{\phi(\omega) = \mathrm{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{2\omega}{5 - 2\omega^2}\right) \text{ in Q1 or Q2.}}$

So, $\boxed{x_p(t) = \dfrac{\omega\cos(\omega t - \phi(w))}{\sqrt{(5 - 2\omega^2)^2 + 4\omega^2}}.}$

(d) (5) *Find the periodic solution to* $P(D)x = \sum_{n=1}^{100} n\cos(nt)$.

Solution: Use Part (c) and superposition: $\boxed{x_p(t) = \sum_{n=1}^{100} \frac{n\cos(nt - \phi(n))}{|P(in)|}}$, where

$\boxed{|P(in)| = \sqrt{(5-2n^2)^2 + 4n^2}}$ and $\boxed{\phi(n) = \tan^{-1}\left(\frac{2n}{5-2n^2}\right) \begin{cases} \text{in Q1} & \text{if } n = 1 \\ \text{in Q2} & \text{if } n > 1 \end{cases}}$

(e) (5) *Find the general real-valued solution to* $(D - 3I)(2D^2 + 2D + 5I)x = 0$;

Solution: Keep this in factored form! The second factor is the characteristic polynomial from Part (a). So the characteristic roots are $-1/2 \pm 3i/2$ and 3. Therefore, the general real-valued solution is $\boxed{x(t) = c_1 e^{-t/2}\cos(3t/2) + c_2 e^{-t/2}\sin(3t/2) + c_3 e^{3t}}$

(f) (10) *Find a particular solution to* $x'' + 9x = \cos(3t)$

Solution: $P(r) = r^2 + 9$. So, $P(3i) = 0$.

$P'(r) = 2r$. So, $P'(3i) = 6i$: $|P'(3i)| = 3$, $\mathrm{Arg}(P'(3i)) = \pi/2$.

By the extended SRF, $\boxed{x_p(t) = \frac{t\cos(3t - \pi/2)}{6} = \frac{t\sin(3t)}{6}.}$

Problem 2. (10 points)
Find all the roots of the equation $r^4 = -16$. *Give them in the form* $a + bi$.

Use this to give the general real-valued solution of $(D^4 + 16I)x = 0$.

If you can't find the roots, then, for 4 points, you can solve the DE pretending the roots are $-2 \pm 3i$, $-4 \pm 5i$.

Solution: In polar form $-16 = 16e^{i(\pi + 2n\pi)}$, where n is any integer.

So, $r^4 = -16 = 16e^{i(\pi+2n\pi)}$. Taking fourth roots, we get,

$$\begin{aligned} r &= 2e^{i\pi/4},\ 2e^{i3\pi/4},\ 2e^{i5\pi/4},\ 2e^{i7\pi/4} \\ &= \sqrt{2} + \sqrt{2}\,i, \quad -\sqrt{2} + \sqrt{2}\,i, \quad -\sqrt{2} - \sqrt{2}\,i, \quad \sqrt{2} - \sqrt{2}\,i \\ &= \sqrt{2} \pm \sqrt{2}\,i, \quad -\sqrt{2} \pm \sqrt{2}\,i. \end{aligned}$$

Using the roots, the general solution is

$$x(t) = c_1 e^{\sqrt{2}\,t}\cos\left(\sqrt{2}\,t\right) + c_2 e^{\sqrt{2}\,t}\sin\left(\sqrt{2}\,t\right) + c_3 e^{-\sqrt{2}\,t}\cos\left(\sqrt{2}\,t\right) + c_4 e^{-\sqrt{2}\,t}\sin\left(\sqrt{2}\,t\right).$$

Problem 3. (10 points)
Consider the equation $x'' + 2x' + kx = 0$.

(a) (5) *Give the range of* k *for which all non-zero solutions are oscillatory.*

Solution: DE: $x'' + 2x' + kx = 0$ (linear, 2^{nd} order, CC, homogeneous).

Charactertic roots: $r = \frac{-2 \pm \sqrt{4 - 4k}}{2}$.

Oscillatory behavior happens if there are complex roots $\Leftrightarrow 4 - 4k < 0 \Leftrightarrow \boxed{k > 1.}$

(b) (5) *Give the range of k for which all solutions go to 0 as $t \to \infty$.*

Solution: From Part (a), the roots are $r = \dfrac{-2 \pm \sqrt{4-4k}}{2}$. All the homogeneous solutions decay to 0 means all the roots have negative real part. So, $\boxed{k} > 0$. (Or you could just appeal to our stability criteria for second-order, constant coefficient DEs.)

Problem 4. (10 points)
Express $(D + tI)(D - tI)$ in the form $a(t)D^2 + b(t)D + c(t)I$. (That is, your answer should have a form like $tD^2 + (t+1)^2 D + (t-1)^2 I$.)

(Hint, apply this operator to an arbitrary test function f.)

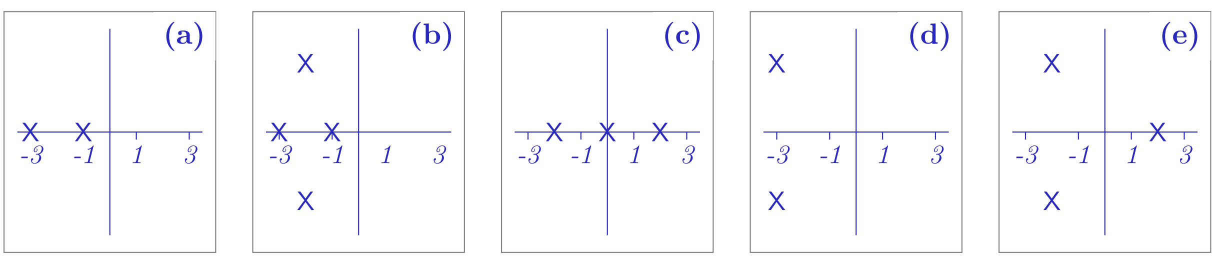
Solution: We apply the operator to an arbitrary test function f. Remembering the product rule, we get:

$$(D + tI)(D - tI)f = (D + tI)(f' - tf) = f'' - f - tf' + tf' - t^2 f = f'' - (t^2 + 1)f$$

That is, $\boxed{(D + tI)(D - tI) = D^2 - (t^2 + 1)I}$.

Problem 5. (10 points)
Each of the pole diagrams below are in the complex plane and the crosses give the characteristic roots of an equation $P(D)x = 0$. (So $P(D)$ is different for each diagram.)

For each part of this problem, you must give a short explanation for your answer.



(a) (4)*List the plot(s) which represent* stable *systems.*

Solution: Stable systems: a, b, d. (All roots must have negative real part.)

(b) (3)*Plot (a) represents a (second-order) damped harmonic oscillator. Is the oscillator under, over or critically damped?*

Solution: Real roots implies overdamped.

(c) (3) *Which of the systems decays to 0 the fastest? You must give a short explanation.*

Solution: Fastest decay: System d (right-most root has most negative real part).

End of quiz solutions