ES.1803 Problem Section Problems for Quiz 2, Spring 2024

Topic 4: Complex numbers and exponentials

Problem 4.1. Make up and solve some simple algebra problems involving addition, subtraction, division, magnitude, complex conjugation.

Problem 4.2. (Polar coordinates)
Write $z = -1 + \sqrt{3}i$ in polar form.

Problem 4.3. (Polar coordinates)
We know $-1 + \sqrt{3}\, i = 2e^{i2\pi/3}$. Use this to answer the following questions.

(a) Compute the product $(-1 + \sqrt{3}i)(a + bi)$ (where a, b are real).

Describe geometrically what multiplying by $-1 + \sqrt{3}i$ does.

(b) What are the polar coordinates of $(-1+\sqrt{3}i)(a+bi)$ in terms of the polar coordinates of $a + bi = re^{i\theta}$?

(c) Describe the sequence of powers of $-1 + \sqrt{3}i$, positive and negative.

Problem 4.4. Write $3e^{i\pi/6}$ in rectangular coordinates.

Problem 4.5. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Problem 4.6. Compute $\dfrac{1}{-2 + 3i}$ in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.

Problem 4.7. Find a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Problem 4.8. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Problem 4.9. (Complex replacement or complexification)
Compute $I = \displaystyle\int e^{4x} \cos(3x)\, dx$ using complex techniques.

Problem 4.10. The point of this problem is to help you distinguish between taking the real part of a function and finding which members of a *family of functions* are real-valued.

(a) Show the inverse Euler formulas are true:

$$\cos(t) = (e^{it} + e^{-it})/2, \qquad \sin(t) = (e^{it} - e^{-it})/2i.$$

(b) Find all the real-valued functions of the form $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are complex constants.

Problem 4.11. Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.

Problem 4.12. Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.

Problem 4.13. Express in the form $a + bi$ the 6 sixth roots of 1.

Problem 4.14. Use Euler's formula to derive the trig addition formulas for sin and cos.

Problem 4.15. Using the polar form, explain why $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z)$ for n a positive integer.

Problem 4.16. Suppose $z^n = 1$. What must $|z|$ be? What are the possible values of $\arg(z)$, if $z^n = 1$?

Problem 4.17. Find the cube roots of i.

Problem 4.18. By using $(e^{it})^4 = e^{4it}$ and Euler's formula, find an expression for $\sin(4t)$ in terms of powers of $\cos(t)$ and $\sin(t)$,

Problem 4.19. Trajectories of $e^{(a+bi)t}$ can vary a lot, depending upon the value of the complex number $a + bi$. The "Complex Exponential" Mathlet shows this clearly. Invoke this applet if you can: `https://mathlets.org/mathlets/complex-exponential/`. You can use it to gain insight into the following questions.

(a) Sketch the trajectory of the complex-valued function $e^{(-1+2\pi i)t}$, and the graphs of its real and imaginary parts.

(b) For each of the following shapes, decide on all the values of $a + bi$ for which the trajectory of $e^{(a+bi)t}$ has this shape.

(i) A circle centered at 0, traversed counterclockwise. What circles are possible?

(ii) A circle centered at 0, traversed clockwise.

(iii) A ray (straight half line) heading away from the origin.

(iv) A curve heading to zero as $t \to \infty$.

Problem 4.20. **(a)** Write $\cos(\pi t) - \sqrt{3}\sin(\pi t)$ in the form $A\cos(\omega t - \phi)$.

(b) Write $5\cos\left(3t + \frac{3\pi}{4}\right)$ in the form $a\cos(\omega t) + b\sin(\omega t)$.

(In each case, begin by drawing a right triangle with sides a and b, angle ϕ, hypotenuse A.)

Problem 4.21. Write $\cos(2t) + \sin(2t)$ in the form $A\cos(\omega t - \phi)$.

Topic 5: Linear, constant coefficient, homogeneous DEs

Problem 5.22. **(a)** Solve $x'' - 8x' + 7x = 0$ using the characteristic equation method.

(b) Solve $x'' + 2x' + 5x = 0$ using the characteristic equation method.

(c) Assume the polynomial $r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$ has roots

$$0.5, \quad 1, \quad 1, \quad 2 \pm 3i.$$

Give the general real-valued solution to the homogeneous constant coefficient DE

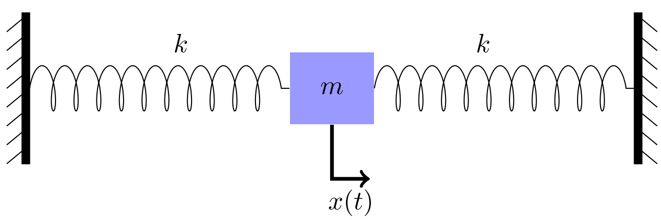
$$x^{(5)} + a_4 x^{(4)} + a_3 x^{(3)} + a_2 x'' + a_1 x' + a_0 x = 0.$$

Problem 5.23. **(Unforced second-order physical systems)**
The DE $x'' + bx' + 4x = 0$ models a damped harmonic oscillator. For each of the values $b = 0, 1, 4, 5$ say whether the system is undamped, underdamped, critically damped or overdamped.

Sketch a graph of the response of each system with initial condition $x(0) = 1$ and $x'(0) = 0$. (It is not necessary to find exact solutions to do the sketch.)

Say whether each system is oscillatory or non-oscillatory.

Problem 5.24. In the spring system below, both springs are unstretched when the position of the mass is $x = 0$, which is exactly in the middle. Write down a DE modeling the position of the mass over time.



Problem 5.25. State and verify the superposition principle for $mx'' + bx' + kx = 0$, (m, b, k constants).

Problem 5.26. A constant coefficient, linear, homogeneous DE has characteristic roots

$$-1 \pm 2i, \ -2, \ -2, -3 \pm 4i.$$

(a) What is the order of the DE? (Notice the $\pm$ in the list of roots.)

(b) What is the general, real-valued solution.

(c) Draw the pole diagram for this system. Explain why it shows that all solutions decay exponentially to 0. What is the exponential decay rate of the general solution?

Topic 6: Operators, inhomogeneous DEs, ERF and SRF

Problem 6.27. Let $P(D) = D^2 + 6D + 5I$. Find the general real-valued solution to each of the following.

(a) $P(D)x = e^{-2t}$.

(b) $P(D)x = \cos(3t)$.

(c) $P(D)x = e^{2t}\cos(3t)$.

(d) $P(D)x = e^{-t}$.

Problem 6.28. (Sinusoidal response formula (SRF))
Let $P(D) = D^2 + 4D + 3$. Find a solution to $P(D)x = \cos(2t)$

Problem 6.29. Solve $P(D)x = x'' + 4x' + 5x = e^{-t}\cos 2t$.

Do this using complex replacement. Give the general solution.

Problem 6.30. Let $P(D) = D^2 + 4D + 6I$. Solve $P(D)x = \cos(2t)$.

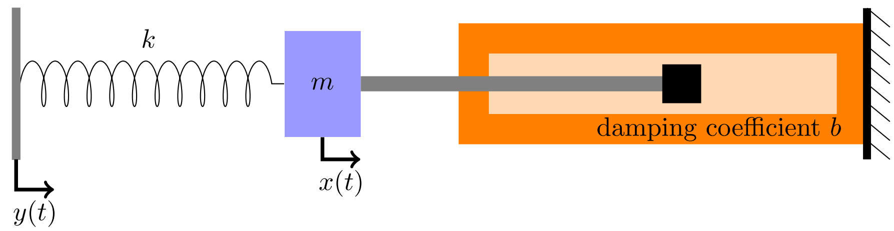
Problem 6.31. **(a)** Solve $x'' + 4x = \cos(\omega t)$ for all possible values of ω.

(b) Plot the graph of your particular solution for $\omega = 2$.

Problem 6.32. **(a)** Show directly from the definition that $P(D) = D^3 + 6D^2 + 7I$ is a linear operator.

(b) Say to yourself: "Checking linearity is always easy. You just have to remember to ask."

Problem 6.33. Driving through the spring. Suppose the spring-mass-dashpot is driven by a mechanism that positions the end of the spring at $y(t)$ as shown. As before, $x(t)$ is the position of the mass. We calibrate x and y so that $x = 0$, $y = 0$ is an equilibrium position of the system.



Give the DE modeling the position $x(t)$ of the mass. Assume, m, k, b, x, y are in compatible units.

Topic 7: Undetermined coefficients for polynomial input

Problem 7.34. (Example from Topic 7 notes.)
Solve $y'' + 5y' + 4y = 2t + 3$ by the method of undetermined coefficients.

Problem 7.35. Solve $x' + 3x = t^2 + t$.

Problem 7.36. Find a particular solution to $x'' + x' = t^4$.
Write down the system of equations for A, B, C, D, E, but don't bother solving.

Topic 8: Applications: stability

Problem 8.37. Is the system $x'' + x' + 4x = 0$ stable?

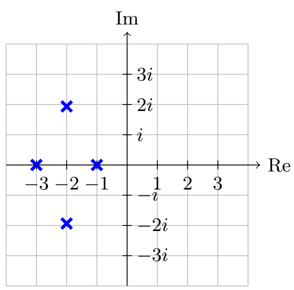
Problem 8.38. Is a 4th order system with roots $\pm 1, -2 \pm 3i$ stable. Which solutions to the homogeneous DE go to 0 as $t \to \infty$?

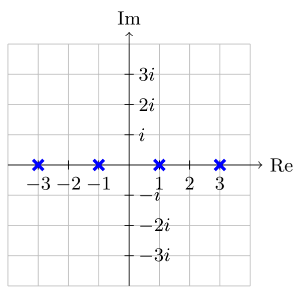
Problem 8.39. For what k is the system $x' + kx = 0$ stable?

Problem 8.40. Consider the following systems.
(i) $x'' + x' + 4x = 0$
(ii) A fourth-order system with roots ± 1, $-2 \pm 3i$
(iii) $x' + 3x = 0$.

Draw the pole diagram for each of these systems and say how it relates to the stability of the system.

Problem 8.41. **(a)** The pole diagram below on the left shows the characteristic roots of the system $P(D)x = 0$.





Left: pole diagram for Part (a). Right: diagram for Part (b)

(i) What is the order of the system?

(ii) Is the system stable?

(iii) Is the system oscillatory?

(iv) What is the exponential decay rate for the general solution?

(b) Repeat Part (a) for the pole diagram on the right.
