ES.1803 Quiz 3 Solutions, Spring 2024

Problem 1. (30 points)
Consider the system $2x'' + 5x' + 8x = \omega\cos(\omega t)$, where we consider $\cos(\omega t)$ to be the input.

(a) (15) *Find the sinusoidal response, gain and phase lag of this system.*

Solution: $P(iw) = 8 - 2\omega^2 + 5i\omega$, so

$$|P(i\omega)| = \sqrt{(8-2\omega^2)^2 + (5\omega)^2}, \quad \phi(\omega) = \arg(P(i\omega)) = \tan^{-1}\left(\frac{5\omega}{8-2\omega^2}\right) \text{ in Q1 or Q2.}$$

The SRF gives us $x_p(t) = \dfrac{\omega}{|P(i\omega)|}\cos(\omega t - \phi(\omega)) = \dfrac{\omega}{\sqrt{(8-2\omega^2)^2+25\omega^2}}\cos(\omega t - \phi(\omega))$.
Therefore, the gain is

$$\boxed{g(\omega) = \frac{\omega}{|P(i\omega)|} = \frac{\omega}{\sqrt{(8-2\omega^2)^2+25\omega^2}}.}$$

and the phase lag is $\phi(\omega)$.

(b) (7) *Find all the practical resonant frequencies of this system.*

Solution: We can do this without much trouble by solving $g'(\omega) = 0$. In this case, there is a little bit of algebra that makes this even easier.

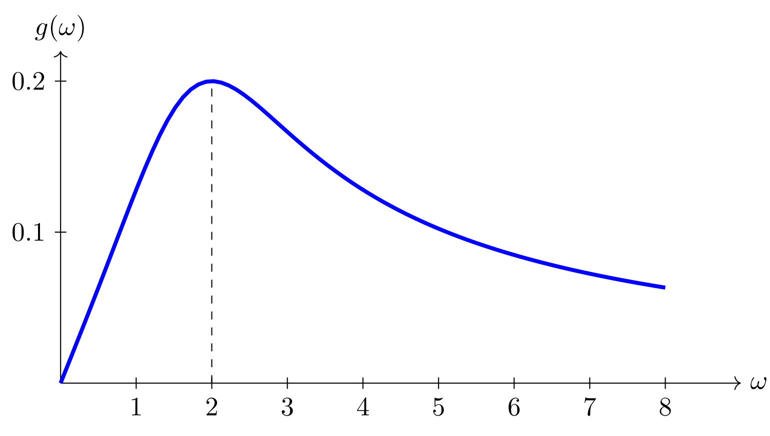
$$h(\omega) = \frac{1}{g(\omega)^2} = \frac{(8-2\omega^2)^2 + 64\omega^2}{\omega^2} = \left(\frac{8}{\omega} - 2\omega\right)^2 + 64$$

The relative maxima of $g(\omega)$ occur at exactly the same place as the relative minima of $h(\omega)$. It is easy to check that $h'(\omega) = 0$ only when $\omega = 2$.

Thus $\boxed{\omega = 2}$ is the only resonant frequency of this system.

(c) (5) *Plot the gain of this system.*

Solution: We can check that $g(0) = 0$ and $g(2) = 0.2$. Since $\omega = 2$ is the only relative maximum, the graph must increase from 0 to 2 and decrease after that.



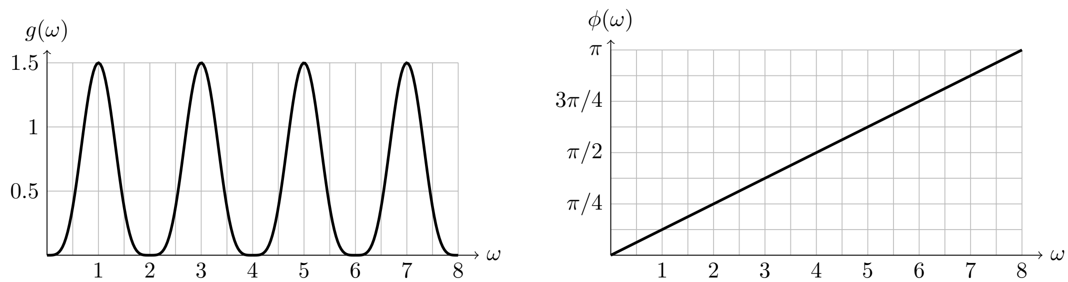
(d) (3) *This system models a damped harmonic oscillator. Find the natural frequency of the oscillator.*

Solution: To find the natural frequency, we set the damping to 0 and find the frequency of the unforced system. That is, solve $2x'' + 8x = 0$. The natural frequency is $\omega = 2$.

Problem 2. (15 points)
(a) (10) *Suppose we have a stable linear constant coefficient system $P(D)x = f$, where we consider f to be the input. The plots of gain $g(\omega)$ and phase lag $\phi(\omega)$ are shown below. Give the periodic solution to the following DE.*

$$P(D)x = \cos(t) + \cos(2t) + \cos(3t).$$

You must give a brief explanation of your reasoning.



Solution: Using the superposition principle, the periodic solution to the DE is

$$x_p(t) = g(1)\cos(t - \phi(1)) + g(2)\cos(2t - \phi(2)) + g(3)cos(3t - \phi(3)).$$

The gain graph shows $g(1) = 1.5$, $g(2) = 0$, $g(3) = 1.5$. The phase graph shows $\phi(1) = \pi/8$ and $\phi(3) = 3\pi/8$ (No need for $\phi(2)$.) So,

$$\boxed{x_p = 1.5\cos(t - \pi/8) + 1.5\cos(3t - 3\pi/8).}$$

(b) (5) *What are the resonant frequencies of this system?*

Solution: Resonances occure at the peaks of the gain graph. The gain graph shows resonance peaks at $w = 1, 3, 5, 7$. We can't know for sure, but we might guess they continue to occur at odd integers.

Problem 3. (15 points)
(a) (10) *Find one solution to the equation $x'' + 9x = \cos(3t)$.*

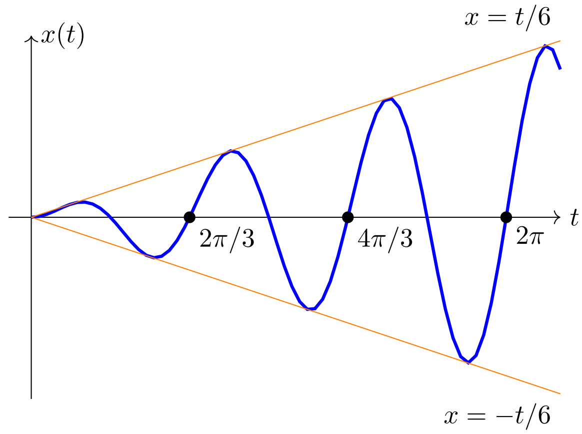
Solution: $P(3i) = 0$, so we need to use the extended SRF.

$P(r) = r^2 + 9$, so $P'(r) = 2r$. $P'(3i) = 6i$, so $|P'(3i)| = 3$, $\text{Arg}(P'(3i)) = \pi/2$. Thus,

$$\boxed{x_p(t) = \frac{t\cos(3t - \pi/2)}{6}} = \frac{t\sin(3t)}{6}.$$

(b) (5) *Draw a graph of the solution from Part (a).*

Solution: The graph is a sine wave with increasing amplitude.



Problem 4. (10 points)
(a) (5) *Consider the system $x'' + 5x' - 6x = f(t)$, where we consider $f(t)$ to be the input. Explain why we would not talk about the gain for this system.*

Solution: The characteristic roots are $r = -6, 1$. Since there is a positive root, the system is not stable. This means the homogeneous solutions will go to infinity. So it is not really relevant to worry about the amplitude of the sinusoidal part of the general solution.

(b) (5) *Explain why a system $P(D)x = \cos(\omega t)$ that has a pure resonant frequency also has a sinusoidal solution to its homogeneous equation $P(D)x = 0$.*

Solution: Pure resonance means that $P(i\omega_0) = 0$. This is exactly the same as saying $\pm i\omega_0$ are roots of the characteristic polynomial. That is, pure resonance can only happen if the characteristic polynomial has pure imaginary roots. Pure imaginary roots imply $C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$ is a solution to the homogeneous DE. (Note, there might be other roots and more solutions.)

End of quiz solutions