ES.1803 Problem Section Problems for Quiz 4, Spring 2024 Solutions

Topic 13: Linearity, matrix multiplication, systems of equations, DEs.

Problem 13.1. *Compute the following by thinking of matrix multiplication as a linear combination of the columns of the matrix.*

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Solution: Picks out second column $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

Solution: Linear combination: $-2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Problem 13.2. *Make up a block matrix problem: Multiply a 4×4 matrix made up of four 2×2 blocks (two blocks of 0s, one block = identity, one block something else) times a 4×2 matrix with (i.e., two 2×2 blocks)*

Solution: You do this!

Problem 13.3. *Is it a vector space? For all of these you just have to check that they are closed under addition and scalar multiplication, i.e. closed under linear combinations.*

(a) *The set of functions $f(x)$ such that $f(5) = 0$.*

Solution: Yes: If $f(5) = 0$ and $g(5) = 0$, then clearly $c_1 f(5) + c_2 g(5) = 0$.

(b) *The set of functions $f(x)$ such that $f(5) = 2$.*

Solution: No: If $f(5) = 2$ and $g(5) = 2$, then $f(5) + g(5) = 4$. So the set is not closed under addition.

(c) *The set of vectors (x, y) in the plane, such that $2x + 3y = 0$.*

Solution: Yes. Graph this, it's a line through the origin. Or, algebraically, if (x_1, y_1) and (x_2, y_2) are in the set, then so is $c_1(x_1, y_1) + c_2(x_2, y_2) = (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$. This is easy to check:

$$2(c_1x_1 + c_2x_2) + 3(c_1y_1 + c_2y_2) = c_1 \underbrace{(2x_1 + 3y_1)}_{\text{equals } 0} + c_2 \underbrace{(2x_2 + 3y_2)}_{\text{equals } 0} = 0$$

(d) *The set of vectors (x, y) in the plane, such that $2x + 3y = 2$.*

Solution: No. Graph this, it's a line NOT through the origin, so it is not closed under scalar multiplication by 0.

Problem 13.4. *Convert the following ODE to a companion system:* $x''' + 2x'' + 3x' + 4x = \cos(5t)$.

Solution: Let $y = x'$, $z = x''$. So, $z' + 2z + 3y + 4x = \cos(5t)$. We get $z' = -4x - 3y - 2z + \cos(5t)$. So,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(5t) \end{bmatrix}$$

Topic 14: Linear algebra: row reduction and subspaces

Problem 14.5. *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 3 & 6 & 10 & 3 & 6 \end{bmatrix}$. *Put A in row reduced echelon form. Find the rank, a basis of the column space, a basis of the null space, and the dimension of each of the spaces.*

Solution: Here are the row reduction steps:

$$A \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The pivot columns are Columns 1 and 3. These give a basis for the column space of A.

$$\text{Basis: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\} \qquad \text{Col}(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\}$$

Note the semantic distinction: the basis set contains 2 vectors, the column space is a set with infinitely many vectors.

Rank(A) = # of pivots = dimension of column space = 2.

The null space of A has dimension 3 = # of free variables. Since A and R have the same null space, we work with R.

We find a basis two ways. First, we solve $R\mathbf{x} = \mathbf{0}$ by writing matrix multiplication as a linear combination of columns.

$$R\mathbf{x} = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_4 + 2x_5 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving for the pivot variables, these equations show, $x_1 = -2x_2 - x_4 - 2x_5$ and $x_3 = 0$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 - 2x_5 \\ x_2 \\ 0 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives us 3 basis vectors:

$$\mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So,

$$\text{Null}(A) = \text{ span of } \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A faster method, is to set, in turn, each free variable to 1 and the others to 0 and then solve for the pivot variables. We do the computation by putting the values below the RREF matrix R.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1	x_2	x_3	x_4	x_5
-2	1	0	0	0
-1	0	0	1	0
-2	0	0	0	1

This gives the same basis we found with our first method.

Problem 14.6. *Let* $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Suppose R is the row reduced echelon form for A.*

(a) *What is the rank of A?*

Solution: A and R have the same rank. Two pivots in R implies rank $= 2$.

(b) *Find a basis for the null space of A.*

Solution: A and R have the same null space. The second and fourth variables are free. We find a basis by setting them to 1 and 0 in turn and then solving for the pivot variables.

Note, I find this computation easiest to do if I think of matrix multiplication as a combination of the columns

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

We show the computation by putting the variables below the matrix. Each row below the matrix shows one solution to $R\mathbf{x} = \mathbf{0}$, found by setting one free variable to 1, the other free variables to 0 and solving for the pivot variables.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -1 & 1 \end{matrix}$$

So a basis of Null(A) contains the two vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

There are, of course, many other bases. Our standard algorithm produces the one given.

(c) *Suppose the column space of A has basis* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. *Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.*

Solution: Looking at R the Columns 1 and 3 are pivot columns. We put the given basis in those columns:

$$A = \begin{bmatrix} 1 & * & 3 & * \\ 1 & * & 1 & * \\ 0 & * & 1 & * \end{bmatrix}$$

The free columns of R are linear combinations of the pivot columns and those of A are the same linear combinations. In R it is clear that

$$\text{Col}_2 = 2 \times \text{Col}_1 \text{ and } \text{Col}_4 = 3 \times \text{Col}_1 + \text{Col}_3.$$

So,

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(d) *Find a matrix with the same row reduced echelon form, but such that* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *are in its column space.*

Solution: We found the relationships between the columns in Part (c). So we put the given columns as pivot columns and construct the free columns from these relationships:

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

Note: you could put any other basis for the subspace generated by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the pivot columns and adjust the free columns accordingly.

Problem 14.7. **(a)** *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & x \end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

Solution: Each column of the product is a multiple of the column vector in the first factor. The 1s show that they are the same multiple. So x must be 2.

Alternatively, each row of the product is a multiple of the row vector. The first column of the product shows that the second row must be twice the first row. So x must be 2.

One equation that works for $x = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

None of the $\bullet$'s can be 0, since that would make the corresponding row or column in the product $\mathbf{0}$.

(b) *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet & 3 \\ \bullet & 4 \\ \bullet & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the $\bullet$'s?

Solution: The matrix equation says: the first column plus twice the second column is zero. So the first column must be $\begin{bmatrix} -6 \\ -8 \\ -10 \end{bmatrix}$.

(c) *Suppose we have a matrix equation*

$$\begin{bmatrix} x & 3 \\ y & 4 \\ z & 5 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Solution: The equation says that the columns of the matrix form a linearly dependent set. That is: one is a multiple of the other. Since the second column is nonzero, we can be sure that the first is a multiple of the second: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ for some t.

Problem 14.8. *Suppose we have a matrix equation*

$$\begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ *is that* $\mathbf{c} \neq \mathbf{0}$. *What can we say about* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$*?*

Solution: To have a nontrivial null space the rank must be less than 3. Since the first and third columns are independent, the middle column must be a linear combination of them. Geometrically, the middle column is in the plane containing the origin and the other two columns.

Problem 14.9. *For what values of* y *is it the case that the columns of* $\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix}$ *form a linearly independent set?*

Solution: The columns are linearly independent when the matrix has rank 3. We can find the rank by row reduction:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & y-3 & -2 \\ 0 & -4 & -4 \end{bmatrix} \xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -4 \\ 0 & y-3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 = -R_2/4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & y-3 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 - (y-3)R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-y \end{bmatrix}$$

If $1 - y \neq 0$, then we have 3 pivots. So the columns are linearly independent exactly when $y \neq 1$.

Problem 14.10. *For the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$*:*

(a) *Find the row reduced echelon form of* A*; call it* R.

Solution: Here are the row reduction steps:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow[R_3 = R_3 + 3R_2]{R_2 = -R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

(b) *The last column of* R *should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of* R. *Find a vector* $\mathbf{x}$, *such that* $R\mathbf{x} = \mathbf{0}$, *which expresses this linear relationship.*

Solution: This is just an awkward way of asking about null vectors. The first two columns are pivotal and the third is free. So the third is a combination of the other two. By inspection of R, we see that

$$\text{Col}_3 = -\text{Col}_1 + 2\text{Col}_2 \quad \Rightarrow \text{Col}_1 - 2\text{Col}_2 + \text{Col}_3 = \mathbf{0}.$$

As a matrix equation this is $R \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) *Verify that the same relationship holds among the columns of A.*

Solution: The third column is indeed minus the first plus twice the second. As a matrix equation,

$$A\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

(d) *Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.*

Solution: Row transformations do the same thing to the entries of all columns.

This is the same as saying that if A and B are related by row-operations, then their null spaces coincide. Their column spaces usually do not.

Problem 14.11. *Suppose we want to solve $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}$.*

(a) *When is this possible? Answer this in the form: "$\mathbf{b}$ must be a linear combination of the two vectors ..."*

Solution: The equation can be solved exactly when $\mathbf{b}$ is a linear combination of the columns of A, i.e., when $\mathbf{b}$ is in Col(A). We can find a basis of Col(A) by finding the RREF. The RREF is

$$R=\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}.$$

The first two columns are pivot columns and the last one is free. Thus, the column space of A is spanned by the first two columns of A. Since, $\mathbf{b}$ must be in Col(A), we can answer the question as follows:

$\mathbf{b}$ must be a linear combination of the first two columns of A, i.e., $\mathbf{b}=c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}2\\3\\4\end{bmatrix}$

Note. Since there are lots of other possible bases for the column space, this is just one of many possible answers.

(b) *$A\mathbf{x}=\mathbf{b}$ is certainly solvable for $\mathbf{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x}=\mathbf{x_p}+\mathbf{x_h}$.*

Solution: $\mathbf{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ is the first column of A, so the obvious solution is $\mathbf{x}=\begin{bmatrix}1\\0\\0\end{bmatrix}$. We can take this (or any other solution!) as $\mathbf{x_p}$. To get the general solution, we must add the general homogeneous solution.

The homogeneous solution is the same as the null space of A or R. We can find that by setting the free variable $x_3 = 1$ and solving for $x_1 = 1$ and $x_2 = -2$. So, $\mathbf{x_h} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+c \\ -2c \\ c \end{bmatrix}.$$

Problem 14.12. *Suppose that the row reduced echelon form of the 4×6 matrix B is*

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) *Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.*

Solution: This is just another way of asking for a basis of Null(B). The null space of B is the same as the null space of its row-echelon form.

The free variables are: x_1, x_3 and x_6. As usual, we set them to 1 in turn.

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1	x_2	x_3	x_4	x_5	x_6
1	0	0	0	0	0
0	−3	1	0	0	0
0	−5	0	−7	−9	1

Thus a basis of Null(B) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ -7 \\ -9 \\ 1 \end{bmatrix} \right\}$$

(b) *Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.*

Solution: $\mathbf{b_1}$ must be $\mathbf{0}$, since applying row operations to it give $\mathbf{0}$, and row operations are reversible.

$\mathbf{b_2} \neq \mathbf{0}$. This is all we can say.

The linear relations among the columns of B are the linear relations among the columns of R: so the columns of B corresponding to the pivot columns of R are independent of the previous columns: $\mathbf{b_2}$, $\mathbf{b_4}$, and $\mathbf{b_5}$.

For the linear relations, we just copy what we know for R: $\mathbf{b_1} = 0$; $\mathbf{b_3} = 3\mathbf{b_2}$; $\mathbf{b_6} = 5\mathbf{b_2} + 7\mathbf{b_4} + 9\mathbf{b_5}$.

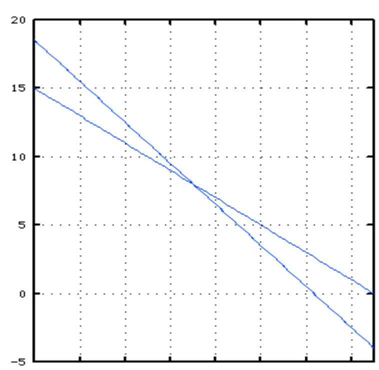
By a 'linearly independent set of vectors of which every vector in the column space is a linear combination' we just mean a basis of Col(B), These are the pivot columns. That is, $\{\mathbf{b_2}, \mathbf{b_4}, \mathbf{b_5}\}$ is a basis for the column space of B.

Problem 14.13. *Solve this system of linear equations. How many methods can you think of to solve this system?*

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Solution: Some ideas:
(1) Graphically with interesecting lines.
(2) Elimination.
(3) Row reduce the augmented matrix.
(4) Matrix inverse.

(1) $y = -x + 5$ and $y = \frac{7}{2} - \frac{3}{2}x$ are two straight lines of different slopes; so they meet at a single point. To find where, we could eyeball the picture—maybe $(-3, 8)$? That satisfies both equations!



(2) We can use elimination: Subtract 3 times the first equation from the second. Retaining the first equation as well, we get

$$\begin{aligned} x + y &= 5 \\ 0 - y &= -8 \end{aligned}$$

and then the first equation gives $x = -3$. In fact, as a second step, we could add the new second equation to the first one:

$$\begin{aligned} x + 0 &= -3 \\ 0 - y &= -8 \end{aligned}$$

Thus $(x, y) = (-3, 8)$ is the solution.

(3) Matrix methods: The system is $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

Again $(x, y) = (-3, 8)$ is the solution.

Problem 14.14. *Consider the following system of equations:*

$$\begin{aligned} x + \ \ y + \ \ z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) *Write this system of equations as a matrix equation.*

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

(b) *Use row reduction to get to row echelon form. What is the solution set?*

Solution: Set up the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \\ 1 & 3 & 6 & 11 \end{array}\right]$$

Do row reduction to RREF

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \\ 1 & 3 & 6 & 11 \end{array}\right] \xrightarrow[\text{Row}_3 = \text{Row}_3 - \text{Row}_1]{\text{Row}_2 = \text{Row}_2 - \text{Row}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 5 & 6 \end{array}\right] \xrightarrow{\text{Row}_3 = \text{Row}_3 - 2\text{Row}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[\text{Row}_1 = \text{Row}_1 - \text{Row}_3]{\text{Row}_2 = \text{Row}_2 - 2\text{Row}_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{\text{Row}_1 = \text{Row}_1 - \text{Row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The solution is $x = 5$, $y = -2$, $z = 2$. You can check this by substituting it into the original equations.

Problem 14.15. *Solve the following equation using row reduction:*

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) *At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?*

Solution: The augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right]$.

Do row reduction:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{\text{Row}_2 = \text{Row}_2 - 3\text{Row}_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -3 \end{array}\right] \xrightarrow{\text{Row}_2 = -\text{Row}_2/3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The last equation now reads $0x + 0y = 1$, which is rather hard to satisfy.

(We could already see this problem after the first reduction step.)

The last column was pivotal. This implies there is a row in the augmented RREF matrix with all zeros except for a 1 in the last column. This row corresponds to the equation $0x + 0y = 1$, which explains why there are no solutions.

(b) *Find a new vector* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *such that* $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *has a solution.*

Solution: Well, we could always take $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because the equation is then obviously solved by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To be more general, we can take $\mathbf{b}$ in the column space of the coefficient matrix. The row reduced echelon form shows that Column 1 is the only pivot column. So the column space has basis $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Thus, the vectors $c \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are exactly the vectors for which the equation admits a solution.

Problem 14.16. *Show that the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *corresponds to counter-clockwise rotation about the origin by 90 degrees, by computing the effect of this matrix on the vectors* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ *and drawing* $\mathbf{v}_1$, $\mathbf{v}_2$, $A\mathbf{v}_1$, $A\mathbf{v}_2$ *on the plane.*

Solution: It's easy to compute:

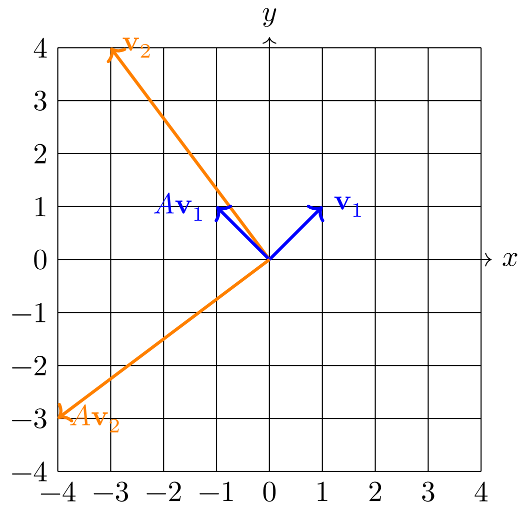
$$A\mathbf{v}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } A\mathbf{v}_2 = \begin{bmatrix} -4 \\ -3 \end{bmatrix}.$$

Using the dot product we can check that $\mathbf{v}_1$ is orthogonal to $A\mathbf{v}_1$:

$$\mathbf{v}_1 \cdot A\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.$$

Similarly $\mathbf{v}_2 \cdot A\mathbf{v_2} = 0$.

This shows hat tA has rotated each of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by 90°. The figure shows the rotation is counter-clockwise.



Topic 15: Transpose, inverse, determinant

Problem 15.17. **(a)** *Use row reduction to find the inverse of the matrix* $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

(b) *Use the record of the row operations to compute the determinant of* A

(a) Solution: Augment the A by the identity and then use row operations to reduce the A to the identity.

$$\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1 \text{ and } R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right]$$
$$\xrightarrow{R_2 - 6R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -7 & 1 & -6 \end{array}\right]$$
$$\xrightarrow{\text{scale } R_2 \text{ by -1/7}} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$
$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2/7 & -5/7 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 2/7 & -5/7 \\ -1/7 & 6/7 \end{bmatrix}$

(b) The only operations that change the determinant are swapping and scaling. In this case, there is one swap and one scale by $-1/7$. The row reduction starts with A and ends with I, so

$$1 = \det(I) = (-1/7) \cdot (-1) \cdot \det(A) \;\Rightarrow\; \boxed{\det(A) = 7}.$$

Problem 15.18. *Use row reduction to find inverses of the following matrices. As you do this, record the row operations carefully for later problems.*

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & 2 & -2 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -6 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row}_3 = \text{Row}_3 3 + 6\text{Row}_1]{\text{Row}_2 = \text{Row}_2 - 2\text{Row}_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & -2 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Row}_3 = \text{Row}_3 - 2\text{Row}_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 10 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Row}_3 = \text{Row}_3/(-2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 1 & -1/2 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 1 & -1/2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & 2 \\ 3 & 5 & 7 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -4 & -8 & -2 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -4 & -8 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

The last row of the reduced matrix has all zeros on the left. This implies the rank of B is 2, and therefore no inverse exists.

(c) $C = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

Solution: No inverse the matrix is not square.

(d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 3 & 2 & 5 \end{bmatrix}$

Solution: Row reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 5 & -1 & 1 & 0 \\ 0 & -4 & -4 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 5 & -1 & 1 & 0 \\ 0 & 0 & -14 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3/(-14)]{R_2 = R_2/(-2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/14 & 1/7 & -1/14 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + \frac{5}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 11/14 & -3/7 & 3/14 \\ 0 & 1 & 0 & 19/28 & -1/7 & -5/28 \\ 0 & 0 & 1 & 1/14 & 1/7 & -1/14 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/7 & -1/7 & 4/7 \\ 0 & 1 & 0 & 19/28 & -1/7 & -5/28 \\ 0 & 0 & 1 & 1/14 & 1/7 & -1/14 \end{array}\right]$$

So $D^{-1} = \begin{bmatrix} -4/7 & -1/7 & 4/7 \\ 19/28 & -1/7 & -5/28 \\ 1/14 & 1/7 & -1/14 \end{bmatrix} = \dfrac{1}{28}\begin{bmatrix} -16 & -4 & 16 \\ 19 & -4 & -5 \\ 2 & 4 & -2 \end{bmatrix}$

Problem 15.19. *Using just the record of the row operations in Problem 15.18 compute the determinant of each matrix.*

(a) Solution: Looking at the effects of the row operations on the $\det(A)$ we get

$R_2 = R_2 - 2R_1$, $R_3R_3 + 6R_1$: leaves determinant unchanged.

$R_3 = R_3 - 2R_2$: leaves determinant unchanged.

$R_3 = R_3/(-2)$: multiplies the determinant by -1/2.

Since $\det(I) = 1$ this gives us $(-1/2)\det(A) = 1$, so $\det(A) = -2$.

(b) Solution: $\det(B) = 0$, because determinant of row reduced form is 0.

(c) Solution: No determinant: the matrix is not square.

(d) Solution: The only row operations that change the determinant are $R_2 = R_2/(-2)$ and $R_3 = R_3/(-14)$. So $(-1/2)(-1/14)\det(D) = 1 \quad \Rightarrow \det(D) = 28$.

Problem 15.20. *Compute the transpose of the following matrices.*

$$A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}. \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Solution: $A^T = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad C^T = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad D^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

Problem 15.21. *Let* $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

Show by direct computation that $(AD)^T = (D^T A^T)$*.*

Solution: $AD = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 31 & 42 & 53 & 64 \\ 11 & 14 & 17 & 20 \end{bmatrix}$ and

$D^T A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 31 & 11 \\ 42 & 14 \\ 53 & 17 \\ 64 & 20 \end{bmatrix}$. Now, by inspection, we see that $(AD)^T = D^T A^T$.

Problem 15.22. **(a)** *Recall the notation for inner product:* $\langle \mathbf{v}, \mathbf{w} \rangle$*. Assume* $\mathbf{v}$ *and* $\mathbf{w}$ *are column vectors. Write the formula for inner product in terms of transpose and matrix multiplication.*

Solution: $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w}$. For example

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = 23.$$

(b) *Using this definition show* $\langle A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A^T \mathbf{w} \rangle$*.*

Solution: $\langle A\mathbf{v}, \mathbf{w} \rangle = (A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T A^T \mathbf{w} = \langle \mathbf{v}, A^T \mathbf{w} \rangle$.

Topic 16: eigenvalues, diagonalization, decoupling

Problem 16.23. *Suppose the* 2×2 *matrix* A *has eigenvectors* $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *with eigenvalues 2 and 4 respectively.*

(a) *Find* $A(\mathbf{v_1} + \mathbf{v_2})$*.*

Solution: $A(\mathbf{v_1} + \mathbf{v_2}) = A\mathbf{v_1} + A\mathbf{v_2} = 2\mathbf{v_1} + 4\mathbf{v_2} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \end{bmatrix}$.

(b) *Find* $A(\mathbf{5v_1} + \mathbf{6v_2})$*.*

Solution: $A(\mathbf{5v_1} + \mathbf{6v_2}) = 5A\mathbf{v_1} + 6A\mathbf{v_2} = 10\mathbf{v_1} + 24\mathbf{v_2} = 10 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 24 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \\ 92 \end{bmatrix}$.

(c) *Find* $A \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

Solution: By inspection or solving some equations, we get $\begin{bmatrix} 4 \\ 9 \end{bmatrix} = 3\mathbf{v_1} + \mathbf{v_2}$. So,

$$A \begin{bmatrix} 4 \\ 9 \end{bmatrix} = 3A\mathbf{v_1} + A\mathbf{v_2} = 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \end{bmatrix}.$$

Problem 16.24. **(a)** *Without calculation, find the eigenvalues and and basic eigenvectors for* $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution: Since $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

Likewise, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3.

(b) *Without calculation, find at least one eigenvector and eigenvalue for* $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Solution: Since $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

The second eigenvector requires a small calculation.

Problem 16.25. **(b)** *Find the eigenvalues and basic eigenvectors of* $A = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.

Solution: Characteristic equation: $\begin{vmatrix} -3-\lambda & 4 \\ 2 & -5-\lambda \end{vmatrix} = \lambda^2 + 8\lambda + 7 = 0 \Rightarrow \lambda = -1, -7.$

Basic eigenvectors for the eigenvalue λ are a basis of $\mathrm{Null}(A - \lambda I)$. That is, basic solutions to $(A - \lambda I)\mathbf{v} = 0$. For the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda_1 = -1$: $(A - \lambda I) = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$. Basic eigenvector: $\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda_2 = -7$: $(A - 7I) = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$. Basic eigenvector: $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Problem 16.26. **(a)** *Find the eigenvalues and basic eigenvectors of* $A = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution: Characteristic equation: $\begin{vmatrix} 3-\lambda & 1 & -3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-2)(\lambda-3) = 0$: eigenvalues are $3, 3, 2$. (Since this is a triangular matrix, you should be able to get these values by inspection.)

The eigenspace corresponding to an eigenvalue λ is $\mathrm{Null}(A - \lambda I)$. We'll need row reduction to find this for each λ.

$$\lambda = 3: \quad (A - 3I) = \begin{bmatrix} 0 & 1 & -3 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\mathrm{Row}_2 = \mathrm{Row}_2 + \mathrm{Row}_1} \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are two free variables (first and third). We use our usual algorithm and notation to find a basis for the null space:

$$\begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{matrix}$$

Our two basis vectors are: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. These are two independent eigenvectors with eigenvalue 3.

For $\lambda = 2$, we won't show the row reduction steps.

$$(A - 2I) = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second column is free. Again, we make our usual computation to find a basis.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 \\ -1 & 1 & 0 \end{matrix}$$

Our basic eigenvector is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(b) *Write A in diagonalized form.*

Solution: Let $\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (diagonal matrix of eigenvalues).

Let $S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (matrix of eigenvectors)

Note: The eigenvectors in S must be in the same order as the eigenvalues in Λ.

We know $A = S\Lambda S^{-1}$. This is the diagonalized form for A.

(c) *Compute A^5.*

Solution: $A^5 = S\Lambda^5 S^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 243 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$

Problem 16.27. *Suppose that the matrix B has eigenvalues 1 and 7, with eigenvectors*

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

respectively.

(a) *What is the solution to* $\mathbf{x}' = Bx$ *with* $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$*?*

Solution: The general solution is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

We use the initial condition to find c_1 and c_2:

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

In matrix form this is $\begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

So, $\mathbf{x}(t) = \frac{1}{3} e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3} e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

(b) *Decouple the system* $\mathbf{x}' = B\mathbf{x}$*. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.*

Solution: Decoupling is just the change of variables $\mathbf{u} = S^{-1}\mathbf{x}$. So,

$$\mathbf{u} = S^{-1}\mathbf{x} \;\Leftrightarrow\; \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \;\Leftrightarrow\; \begin{cases} u = x/6 - 5x/6 \\ v = x/6 + y/6. \end{cases}$$

In these coordinates the decoupled system is $\mathbf{u}' = \Lambda \mathbf{u} \;\Leftrightarrow\; \begin{bmatrix} u' \\ v' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

(c) *Give an argument based on transformations why* $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ *has the eigenvalues and eigenvectors given above.*

Using the definition of eigenvalues and eigenvectors, we need to show that

$$B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Multiplying by a standard basis vector just picks out the corresponding column of a matrix. So we have the following multiplication table:

$$\begin{aligned}
S \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Rightarrow S^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
S \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \Rightarrow S^{-1} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\Lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\Lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Using this table, we can now compute the product $S\Lambda S^{-1}\begin{bmatrix}5\\1\end{bmatrix}$.

$$S\Lambda S^{-1}\begin{bmatrix}5\\1\end{bmatrix} = S\Lambda\begin{bmatrix}0\\1\end{bmatrix} = S\begin{bmatrix}0\\7\end{bmatrix} = 7S\begin{bmatrix}0\\1\end{bmatrix} = 7\begin{bmatrix}5\\1\end{bmatrix}.$$

This shows that $\begin{bmatrix}5\\1\end{bmatrix}$ is an eigenvector of $S\Lambda S^{-1}$ with eigenvalue 7. The other eigenvalue/eigenvector pair behaves the same way.

Problem 16.28. *Suppose* $A = \begin{bmatrix}a & b & c\\0 & 2 & e\\0 & 0 & 3\end{bmatrix}$.

(a) *What are the eigenvalues of A?*

Solution: For an upper triangular matrix the eigenvalues are the diagonal entries: a, 2, 3.

(b) *For what value (or values) of a, b, c, e is A singular (non-invertible)?*

Solution: $\det(A)$ = product of eigenvalues. So A is singular when $a = 0$. The parameters b, c, e can take any values.

(c) *What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?*

Solution: When $a = 0$, the null space is dimension 1, so rank =2.

When $a \neq 0$, A is invertible, so has rank = 3.

(d) *Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.*

Solution: The two positive eigenvalues imply the system is unstable.

Problem 16.29. *Suppose that* $A = S\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}S^{-1}$.

(a) *What are the eigenvalues of A?*

Solution: The eigenvalues are the same as the diagonal matrix, i.e., 1, 2, 3.

(b) *Express A^2 and A^{-1} in terms of S.*

Solution: $A^2 = S\begin{bmatrix}1 & 0 & 0\\0 & 4 & 0\\0 & 0 & 9\end{bmatrix}S^{-1}$; $\quad A^{-1} = S\begin{bmatrix}1 & 0 & 0\\0 & 1/2 & 0\\0 & 0 & 1/3\end{bmatrix}S^{-1}$.

(c) *What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?*

Solution: You need to know the eigenvector that goes with the eigenvalue 3. That is, you need to know the third column of S.

Problem 16.30.

(a) *An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.*

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. *Find an orthogonal matrix* S *and a diagonal matrix* Λ *such that* $A = S\Lambda S^{-1}$

Solution: The problem is asking us to diagonalize A, taking care that the matrix S is orthogonal.

A has characteristic equation: $\lambda^2 - 2\lambda - 3$. So it has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$. By inspection (or computation), we have eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These are clearly orthogonal to each other. We normalize their lengths and use the normalized eigenvectors in the matrix S.

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad \Rightarrow A = S\Lambda S^{-1}.$$

Note: A is a symmetric matrix. It turns out that symmetric matrix has an orthonormal set of basic eigenvectors.

(b) *Decouple the equation* $\mathbf{x}' = A\mathbf{x}$, *with* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution: The decoupling change of variable is $\mathbf{u} = S^{-1}\mathbf{x} \Leftrightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The decoupled system is $\mathbf{u}' = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} \Leftrightarrow \begin{cases} u_1' = -u_1 \\ u_2' = 3u_2 \end{cases}$.

Problem 16.31. *Find the eigenvalues and basic eigenvectors of* $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$.

Solution: Characteristic equation: $\begin{vmatrix} -3-\lambda & 13 \\ -2 & -1-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 29 = 0 \Rightarrow \lambda = -2 \pm 5i$.

Basic eigenvectors for λ are a basis of Null$(A - \lambda I)$.

$\lambda_1 = -2 + 5i$: $(A - \lambda_1 I)\mathbf{v} = \begin{bmatrix} -1-5i & 13 \\ -2 & 1-5i \end{bmatrix} \mathbf{v_1} = 0$. Basic eigenvector: $\mathbf{v_1} = \begin{bmatrix} 13 \\ 1+5i \end{bmatrix}$.

$\lambda_2 = -2 - 5i$: Use complex conjugate: $\mathbf{v_2} = \overline{\mathbf{v}}_\mathbf{1} = \begin{bmatrix} 13 \\ 1-5i \end{bmatrix}$.

Topic 17: Matrix methods of solving systems of DEs

Problem 17.32. **(a)** *Let* $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. *Solve* $\mathbf{x}' = A\mathbf{x}$.

Solution: Characteristic equation: $|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 10 = 0$. So the eigenvalues are $\lambda = 2, -5$.

Basic Eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda = 2$: $\quad A - \lambda I = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$\lambda = -5$: $A - \lambda I = \begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Two (modal) solutions: $\mathbf{x_1}(t) = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{x_2}(t) = e^{-5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

General solution: $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

(b) *What is the solution to $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.*

Solution: We use the initial condition to find values for the parameters c_1, c_2

$$\mathbf{x}(0) = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

In matrix form we have $\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6/7 \\ -4/7 \end{bmatrix}.$$

Thus, $\mathbf{x}(t) = \frac{6}{7}e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{4}{7}e^{-5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(c) *Decouple the system in Part (a). That is, make a change of variables that converts the system to a decoupled one. Write the system in the new variables.*

Solution: The decoupling change of variables is $\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} u \\ v \end{bmatrix}$, where S is the matrix of eigenvectors. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} x &= 3u + v \\ y &= 2u + 3v \end{cases}$$

In these variables the system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \Lambda\begin{bmatrix} u \\ v \end{bmatrix}$, where Λ is the diagonal matrix of eigenvalues. That is,

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} u' &= 2u \\ v' &= -5v \end{cases}$$

Problem 17.33. *Solve $x' = -3x + y$, $y' = 2x - 2y$.*

Solution: The coefficient matrix is $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 5\lambda + 4 = 0$. This has roots $\lambda = -1, -4$.

Basic eigenvectors (basis of $\text{Null}(A - \lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = -1$: $A - \lambda I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. Basic eigenvector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\lambda = -4$: $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Basic eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Modal solutions: $\mathbf{x_1}(t) = e^{-t}\begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{x_2}(t) = e^{-4t}\begin{bmatrix}1\\-1\end{bmatrix}$.

General solution $\mathbf{x}(\mathbf{t}) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) = c_1 e^{-t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-4t}\begin{bmatrix}1\\-1\end{bmatrix}$.

Problem 17.34. ***(Complex roots)*** *Solve* $\mathbf{x}' = \begin{bmatrix}7 & -5\\4 & 3\end{bmatrix}\mathbf{x}$ *for the general real-valued solution.*

Solution: Coefficient matrix: $A = \begin{bmatrix}7 & -5\\4 & 3\end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix}7-\lambda & -5\\4 & 3-\lambda\end{vmatrix} = \lambda^2 - 10\lambda + 41 = 0$.

Eigenvalues: $\lambda = 5 \pm 4i$.

Basic eigenvectors (basis of Null$(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 5+4i$: $A - \lambda I = \begin{bmatrix}2-4i & -5\\4 & -2-4i\end{bmatrix}\mathbf{v} = \begin{bmatrix}0\\0\end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix}5\\2-4i\end{bmatrix}$.

Complex solution:

$$\mathbf{z}(t) = e^{(5+4i)t}\begin{bmatrix}5\\2-4i\end{bmatrix} = e^{5t}\begin{bmatrix}5\cos(4t) + i5\sin(4t)\\2\cos(4t) + 4\sin(4t) + i(-4\cos(4t) + 2\sin(4t))\end{bmatrix}.$$

Both real and imaginary parts are solutions to the DE:

$$\mathbf{x_1}(t) = e^{5t}\begin{bmatrix}5\cos(4t)\\2\cos(4t)+4\sin(4t)\end{bmatrix}, \quad \mathbf{x_2}(t) = e^{5t}\begin{bmatrix}5\sin(4t)\\-4\cos(4t)+2\sin(4t)\end{bmatrix}$$

General real-valued solution (by superposition): $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

Problem 17.35. ***(Repeated roots)*** *Solve* $\mathbf{x}' = \begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix}\mathbf{x}$.

Solution: The coefficient matrix is $A = \begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix}3-\lambda & 1\\-1 & 1-\lambda\end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$

Eigenvalues: $\lambda = 2, 2$ (repeated)

Basic eigenvectors (basis of Null$(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 2$: $A - \lambda I = \begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}$. Basic eigenvector $\mathbf{v_1} = \begin{bmatrix}1\\-1\end{bmatrix}$.

This gives one modal solution: $\mathbf{x_1}(t) = e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$.

Since there are not enough independent eigenvectors, the system is defective. For the second solution, we look for one of the form

$$\mathbf{x_2} = te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\mathbf{w}.$$

($\mathbf{w}$ is called a generalized eigenvector. It satisfies $(A - 2I)\mathbf{w} = \mathbf{v}$.)

After some algebra, we find that we can take $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, $\mathbf{x_2}(t) = te^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

General solution: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

Problem 17.36. *Solve the system $x' = x + 2y$; $y' = -2x + y$.*

Solution: The coefficient matrix is $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0.$

Eigenvalues $1 \pm 2i$.

Basic eigenvectors (basis of $\text{Null}(A - \lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 1 + 2i$: $A - \lambda I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$. Basic eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

(We don't need an eigenvector from the complex conjugate $\lambda = 1 - 2i$.)

Complex solution: $\mathbf{z}(t) = e^{(1+2i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t(\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t\begin{bmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{bmatrix}$.

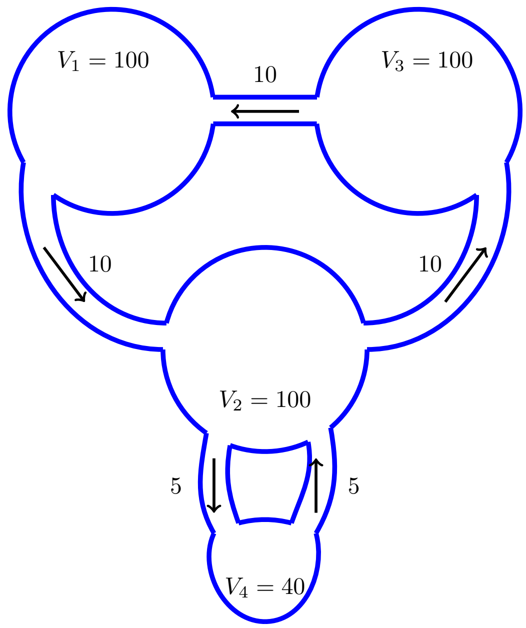
The real and imaginary parts of $\mathbf{z}$ are both solutions:

$\mathbf{x_1}(t) = e^t\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$, $\mathbf{x_2}(t) = e^t\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$.

General solution: $\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1e^t\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2e^t\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$.

Or $x(t) = c_1e^t\cos(2t) + c_2e^t\sin(2t)$; $y(t) = -c_1e^t\sin(2t) + c_2e^t\cos(2t)$.

Problem 17.37. *The following figure shows a closed tank system with volumes and flows as indicated (in compatible units). Let's call the tank with $V_1 = 100$ tank 1, etc.*



(a) *Write down a system of differential equations modeling the amount of solute in each tank.*

Solution: Let x_1, x_2, x_3, x_4 be the amount of solute in tanks 1 to 4 respectively. Note that the system is balanced, in that the volume in each tank is not changing. Using rate = rate in - rate out we get the following equations.

$$\begin{aligned}
x_1' &= -10\frac{x_1}{V_1} + 10\frac{x_3}{V_3} = -0.1x_1 + 0.1x_3 \\
x_2' &= 10\frac{x_1}{V_1} - 15\frac{x_2}{V_2} + 5\frac{x_4}{V_4} = 0.1x_1 + -0.15x_2 + 0.125x_4 \\
x_3' &= 10\frac{x_2}{V_2} - 10\frac{x_3}{V_3} = 0.1x_2 - 0.1x_3 \\
x_4' &= 5\frac{x_2}{V_2} - 5\frac{x_4}{V_4} = 0.05x_2 - 0.125x_4
\end{aligned}$$

In matrix form this is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} -0.1 & 0 & 0.1 & 0 \\ 0.1 & -0.15 & 0 & 0.125 \\ 0 & 0.1 & -0.1 & 0 \\ 0 & 0.05 & 0 & -0.125 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) *Without computation you know one eigenvalue. What is it? What is a corresponding eigenvector?*

Solution: Eventually the system has to reach equilibrium, where all the concentrations are equal. This means one eigenvalue is 0. At equilibrium we must have

$$\frac{x_1}{V_1} = \frac{x_2}{V_2} = \frac{x_3}{V_3} = \frac{x_4}{V_4}.$$

Therefore, using the values for the volumes, we have

$$x_2 = x_1, \quad x_3 = x_1, \quad x_4 = 0.4x_1.$$

We have $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0.4 \end{bmatrix}$ is an eigenvector.

(c) *What can you say about all the other eigenvalues?*

Solution: They all must be negative, or complex with negative real part. If any were positive, the amount of solute would be growing, which is impossible in a closed system.
