

## ES.1803 Quiz 4, Spring 2024

5 problems, No books, notes or calculators.

**Problem 1.** (20 points)

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 7 \\ 2 & 4 & 6 & 14 \end{bmatrix}$$

- (a) (10) Put  $A$  in reduced row echelon form.  
 (b) (5) Give a basis for the column space of  $A$ .  
 (c) (3) What is the dimension of  $\text{Null}(A)$ ?  
 (d) (2) What is the rank of  $A$ ?

**Problem 2.** (20 points)

$$\text{The matrix } R \text{ is in reduced row echelon form: } R = \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- (a) (10) Give the general solution to the equation  $R\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ .

- (b) (10) Find a matrix  $A$  with reduced row echelon form  $R$  and such that the equations

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} \text{ and } A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ can both be solved.}$$

**Problem 3.** (30 points)

- (a) (10) Let  $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ . Find the general *real-valued* solution to  $\mathbf{x}' = A\mathbf{x}$ .

Helpful check: Your eigenvalues should be integers.

- (b) (10) Suppose  $B$  is a  $3 \times 3$  matrix with eigenvalues 3, 7, 10 and corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- (i) Find  $\det(B)$ .      (ii) Find  $B \left( 2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ .      (iii) Find  $B^{-1} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ .

- (iv) Give the general solution to the system of DEs  $\mathbf{x}' = B^2\mathbf{x}$ .

(Parts c and d on next page.)

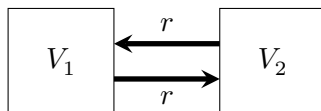
(c) (5) Give the diagonalized form of the matrix  $B$  from Part (b). That is, write it as a product of certain matrices. You do not have to find inverses explicitly.

(d) (5) Suppose  $D$  is a  $2 \times 2$  matrix with eigenvalues  $1 + 2i$  and  $1 - 2i$  and corresponding eigenvectors  $\begin{bmatrix} 1 \\ 3i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -3i \end{bmatrix}$ .

Give the general real-valued solution to the equation  $\mathbf{x}' = D\mathbf{x}$ .

**Problem 4.** (10 points)

Consider the closed two-compartment mixing tank system shown. Let  $x, y$  be the amount of salt in tanks 1, 2 respectively. The volumes  $V_1, V_2$  and the flow rate  $r$  are (positive) constants.



Assume compatible units and write down in matrix form the system of DEs governing the amount of salt in the tanks.

**Problem 5.** (20 points)

Give a short explanation for each answer.

(a) (4) Suppose  $A$  is a square matrix with RREF  $R$ . True or false:  $A$  and  $R$  have the same eigenvalues.

(b) (4) Find the companion system to the DE  $x'' + 2x' + 7x = 0$ . Give your answer in matrix form.

(c) (4) Consider the set of all series of the form  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$ . Here,  $a_0, a_n, b_n$  are parameters that can take arbitrary values. Is this set a vector space?

(d) (4) **True or false:** Suppose  $A$  is a square matrix. If the linear system  $A\mathbf{x} = \mathbf{0}$  has more than one solution, then  $\det A = 0$ .

(e) (4) Suppose  $E$  is a  $2 \times 2$  matrix with eigenvalues 1 and -3 and corresponding eigenvectors  $\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ .

Suppose  $\begin{bmatrix} x \\ y \end{bmatrix}$  is a solution to the system  $\mathbf{x}' = E\mathbf{x}$ . As  $t$  gets large, the ratio of  $x$  to  $y$  goes asymptotically to what value?

*End of quiz*

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