ES.1803 Problem Section Problems for Quiz 5, Spring 2024

Topic 20 Step and delta functions

Problem 20.1. Compute the following integrals.

(a) $\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\, dt$

(b) $\int_{1}^{5} \delta(t) + 3\delta(t-2) + 4\delta(t-6)\, dt.$

Problem 20.2. Compute the following integrals.

(a) $\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\, dt.$

(b) $\int \delta(t)\, dt.$ (Indefinite integral)

(c) $\int \delta(t) - \delta(t-3)\, dt.$ Graph the solution

Problem 20.3. Solve $x' + 2x = \delta(t)$ with rest IC

Problem 20.4. **(a)** Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.

(b) Plug your solution into the DE and verify that it is correct

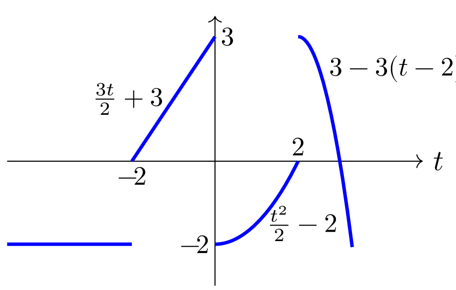
Problem 20.5. Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC

Problem 20.6. (Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.

Problem 20.7. Solve $x' + 3x = \delta(t) + e^{2t}u(t) + 2\delta(t-4)$ with rest IC.

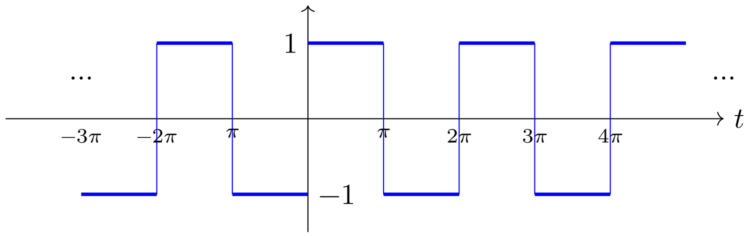
(The $u(t)$ is there to make sure the input is 0 for $t < 0$.)

Problem 20.8. The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.



Problem 20.9. **Derivative of a square wave**

The graph below is of a function sq(t) (called a square wave). Compute and graph its generalized derivative.



Graph of sq(t) = square wave

Topic 21 Fourier series: basics

Integral table

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}$$

$$\int t\sin(\omega t)\,dt = -\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}$$

$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}$$

$$\int t^2\sin(\omega t)\,dt = -\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}$$

$$\int e^t\cos(\omega t)\,dt = \frac{e^t\cos(\omega t)}{1+\omega^2} + \frac{\omega e^t\sin(\omega t)}{1+\omega^2}$$

$$\int e^t\sin(\omega t)\,dt = -\frac{\omega e^t\cos(\omega t)}{1+\omega^2} + \frac{e^t\sin(\omega t)}{1+\omega^2}$$

$$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

$$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

$$\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$$

$$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$

$$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$

$$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$

Problem 21.10. For each of the following:

(i) Find the Fourier series (no integrals needed)

(ii) Identify the fundamental frequency and corresponding base frequency.

(iii) Identify the Fourier coefficients a_n and b_n

(a) $\cos(2t)$

(b) $3\cos(2t - \pi/6)$

(c) $\cos(t) + 2\cos(5t)$

(d) $\cos(3t) + \cos(4t)$

Problem 21.11. Compute the Fourier series for the odd, period 2π, amplitude 1 square wave.

Problem 21.12. Compute the Fourier series for the period 2π triangle wave

$$f(t) = |t| \text{ for } -\pi < t < \pi.$$

Problem 21.13. Consider the period 1 function given by $f(t) = e^t$ on $(0, 1)$.

(a) Graph the function.

(b) What would you expect about the decay rate of the Fourier coefficients?

(c) Compute the Fourier series. The integral table provided might help.

Topic 22: Fourier series: basics continued

Problem 22.14. Let $f(t)$ be the odd, period 2, amplitude 1 square wave. Carefully sketch the graph of the Fourier series.

Problem 22.15. **(a)** Compute the Fourier series for the even, period 2π function, with $f(t) = \pi t - t^2$ on $[0, \pi]$. The integral table provided should help.

(b) Carefully sketch the graph of the Fourier series.

(c) Challenge: Can you explain why the odd cosine coefficients are 0?

Problem 22.16. Recall the Fourier series for the period 2π triangle wave $\text{tri}(t)$, where $\text{tri}(t) = |t|$ for $-\pi \le t \le \pi$:

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}.$$

Set $t = 0$ and show $\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$. (This is only for fun, we will not test on this sort of problem.)

Problem 22.17. The function $f(t)$ has period π. Over the interval $0 \le x < \pi$ we have $f(t) = \sin(t)$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$

Topic 23: Sine and cosine series; calculation tricks

Integral table

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}$$
$$\int t\sin(\omega t)\,dt = -\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}$$
$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}$$
$$\int t^2\sin(\omega t)\,dt = -\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}$$
$$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$
$$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$
$$\int \cos(at)\sin(bt)\,dt = -\frac{1}{2}\left[\frac{\cos((a+b)t)}{a+b} - \frac{\cos((a-b)t)}{a-b}\right]$$
$$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$
$$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$
$$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$

Problem 23.18. Find Fourier cosine series for $\sin(x)$ on $[0,\pi]$.

Problem 23.19. Find the Fourier cosine series for the function $f(x) = x^2$ on $[0,1]$. Graph the function and its even period 2 extension.

Problem 23.20. Find the Fourier series for the standard square wave shifted to the left so it's an even function, i.e., $sq(t+\pi/2)$.

Problem 23.21. Find the Fourier sine series for $f(x) = 1$ on $[0,\pi]$.
