Name ______________________________

ES.1803 Quiz 5, Spring 2024

4 problems, No books, notes or calculators.

Problem 1. (40 points)

(a) (5) Compute $\displaystyle\int_{0^-}^{8^-} t^2\Big(\delta(t)+2\delta(t-4)+4\delta(t-8)+6\delta(t-16)\Big)\,dt.$

(b) (5) Compute $\displaystyle\int e^{2t}\Big(\delta(t)+2\delta(t-3)\Big)\,dt.$

(c) (5) Consider the following function

$$f(t)=\begin{cases}0 & \text{for } t<0\\ 2t & \text{for } 0<t<1\\ (t-1)^2 & \text{for } 1<t<2\\ 0 & \text{for } 2<t\end{cases}$$

Find the generalized derivative of $f(t)$.

(d) (15) Find the solution to the following DE with the given initial conditions.

$$2x''+8x=3\delta(t-\pi);\quad x(0)=0,\ x'(0)=2.$$

(e) (10) Let $g(t)$ be the period 2 function

$$g(t)=\ldots+2\delta(t+4)+2\delta(t+2)+2\delta(t)+2\delta(t-2)+2\delta(t-4)+2\delta(t-6)+\ldots$$

Find the Fourier series of $g(t)$.

Problem 2. (15 points)

The periodic function $f(t)$ has Fourier series $f(t)=3+5\displaystyle\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2+1}$. Find the periodic solution to the DE $3x'+x=f(t)$.

Problem 3. (10 points)
Find the Fourier cosine series for $f(x)=1+x$ on the interval $[0,\pi]$.

Problem 4. (15 points)
(a) (5) Let $f(x)=x^2$ on the interval $[0,1]$.

What is the decay rate for the coefficients of Fourier cosine series for f?

(b) (5) Let $f(t)=\cos(2t)+2\cos(3t)$. Identify the smallest base period and corresponding fundamental angular frequency for f.

(c) (5) True or false: if the function $g(t)$ is periodic and even, then the periodic solution to $3x'+x=g(t)$ is also even. You must give a short justification for your answer.

End of quiz

Integrals (for n a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$

Some Fourier series:

1. Period 2π square wave $\mathrm{sq}(t)$: *You should know this for the quiz.*
2. Period 2 triangle wave $\mathrm{tri2}(t)$:

   Over one period, $-1 \le t \le 1$, $\mathrm{tri2}(t) = |t|$.

$$\begin{aligned}\mathrm{tri2}(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right) \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}.\end{aligned}$$