ES.1803 Quiz 6 Solutions, Spring 2024

Problem 1. (20 points)
(a) (15) *Find the periodic solution to the DE* $x'' + x' + 17x = 1 + \mathrm{sq}(t)$*, where* $\mathrm{sq}(t)$ *is our usual odd, period* 2π*, amplitude 1 square wave.*

Solution: We know $\mathrm{sq}(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$

First solve for each piece (without coefficients) using the SRF:

Constant term: $x_0'' + x_0' + 17x_0 = 1 \Rightarrow x_{0,p} = 1/17$.

Other terms: $x_n'' + x_n' + 17x_n = \sin(nt) \Rightarrow x_{n,p}(t) = \frac{\sin(nt - \phi(n))}{|P(in)|}$,
where $P(in) = 17 - n^2 + in$, so

$$|P(in)| = \sqrt{(17-n^2)^2 + n^2}, \text{ and } \boxed{\phi(n) = \arg(P(in)) = \tan^{-1}\left(\frac{n\pi}{17-n^2}\right) \text{ in Q1 or Q2}}$$

So, by superposition, $\boxed{x_p(t) = \frac{1}{17} + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt - \phi(n))}{n\sqrt{(17-n^2)^2 + n^2}}.}$

(b) (5) *The period* 2π *impulse train has Fourier series* $f(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos(nt)$*. Without solving the DE, say which term in the expansion of* $f(t)$ *causes the biggest response in the system* $x'' + x' + 17x = f(t)$*?*

Solution: This is a lightly damped harmonic oscillator with natural frequency $\sqrt{17} \approx 4$. So the $n = 4$ term causes the biggest response.

Problem 2. (30 points) *For this problem we have a vibrating string of length* π*. The amplitude is given by* $y(x,t)$*, where* x *is the position along the string and* t *is time. The vibration is modeled by the wave equation*

$$\text{PDE: } y_{tt}(x,t) = 9y_{xx}(x,t), \qquad 0 \le x \le \pi, \quad t \ge 0.$$

The ends of the string are set, so that they satisfy the boundary conditions

$$\text{BC: } y_x(0,t) = 0, \quad y_x(\pi,t) = 0.$$

Find the general solution to this PDE with these boundary conditions. You must show all the steps leading to the solution.

Solution: Step 1: Find separated solutions: $y(x,t) = X(x)T(t)$.

Substitution gives $XT'' = 9X''T$, so,

$$\frac{X''}{X} = \frac{T''}{9T} = -\lambda \text{ for some constant } \lambda.$$

(We must have a constant because we have a function of t = a function of x.) Thus we have two ordinary differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad T'' + 9\lambda T = 0.$$

There are 3 cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$, $\quad T(t) = c\cos(3\sqrt{\lambda}\,t) + d\sin(\sqrt{\lambda}\,t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $\quad T(t) = c + dt$

Case (iii) $\lambda < 0$: We ignore this since it never produces nontrivial solutions satisfying the boundary conditions.

Step 2: Find modal solutions (separated solutions which satisfy the BC).

For separated solutions the BC are $X'(0) = 0$, $X'(\pi) = 0$.

Case (i): $X'(0) = b\sqrt{\lambda} = 0$, $X'(\pi) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)$.

The nontrivial solutions must have $a \neq 0$ and $\sqrt{\lambda} = n$, where n is an integer. So we have modal solutions:

$$y_n(x,t) = \cos(nx)(c_n\cos(3nt) + d_n\sin(3nt)), \text{ for } n = 1, 2, \ldots$$

Case (ii): $X'(0) = b = 0$, $X'(\pi) = b = 0$.

So a can take any value and this case gives the modal solution, $y_0(x,t) = \frac{c_0}{2} + \frac{d_0 t}{2}$. (We include the $\frac{1}{2}$ out of habit – it would help if we had IC.)

Step 3: Superpostion. By superposition we have the general solution to the PDE and BC:

$$\boxed{y(x,t) = y_0(x,t) + \sum_{n=1}^{\infty} y_n(x,t) = \frac{c_0}{2} + \frac{d_0 t}{2} + \sum_{n=1}^{\infty}\cos(nx)(c_n\cos(3nt) + d_n\sin(3nt))}.$$

Problem 3. (25 points) *Recall the PDE (heat equation) with BC:*

PDE: $w_t(x,t) = w_{xx}(x,t)$, $\quad t \geq 0$, $\quad 0 \leq x \leq \pi$.

BC: $w(0,t) = 0$, $\quad w(\pi,t) = 0$.

We have seen many times that the general solution to this is $w(x,t) = \sum_{n=1}^{\infty} b_n\sin(nx)e^{-n^2 t}$.

(a) *(10) Now consider the inhomogeneous PDE and BC:*

IPDE: $u_t(x,t) = u_{xx}(x,t) - 2$, $\qquad 0 \leq x \leq \pi$, $\quad t \geq 0$.

IBC: $u(0,t) = 0$, $\quad u(\pi,t) = \pi^2$.

Give the general solution to IPDE and IBC:

Hint: Start by guessing a steady-state (time independent) particular solution.

Solution: Following the hint we guess a solution $u(x,t) = X(x)$. Plugging this into IPDE gives

$$0 = X''(x) - 2.$$

We solve this as follows: $X''(x) = 2 \Rightarrow X(x) = x^2 + ax + b$.

Now we have to find a and b that match the boundary conditions IBC. Plugging in $x = 0$ gives: $X(0) = b = 0$.

Now, plugging in $x = \pi$ gives: $X(\pi) = \pi^2 + a\pi = \pi^2 \longrightarrow a = 0$.

We have found a particular solution to IPDE + IBC: $u_p(x,t) = x^2$.

The function w given above solves the associated homogeneous PDE with homogeneous boundary conditions. So the general solution to IPDE + IBC is

$$u(x,t) = u_p(x,t) + w(x,t) = \boxed{x^2 + \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}.}$$

(b) (10) *Suppose that initially the rod has temperature $u(x,0) = x^2 + x$. Find the solution from Part (a) with these initial conditions.*

Solution: Plugging the IC into the general solution from Part (a) we get:

$$u(x,0) = x^2 + \sum_{n=1}^{\infty} b_n \sin(nx) = x^2 + x, \text{ on } [0,\pi].$$

So, $\sum b_n \sin(nx) = x$, i.e., b_n are the sine coefficients of $f(x) = x$. We can find these using Formula 1′ in the integral table: $\boxed{b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2(-1)^{n+1}}{n}}$.

Thus we have, $\boxed{u(x,t) = x^2 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) e^{-n^2 t}}$.

(c) (5) *Estimate the temperature of the rod at position $x = \pi/2$ at time $t = 1$, i.e., estimate $u(\pi/2, 1)$. You must give a finite numerical expression for the answer. Justify your answer, but you don't need to simplify the expression.*

Solution: Plugging in $x = \pi/2$ and $t = 1$ and writing the first 4 terms of the solution explicitly we get:

$$u(\pi/2,1) = \frac{\pi^2}{4} + 2e^{-1} - \frac{2}{2}\sin(\pi)e^{-4} + \frac{2}{3}\sin(3\pi/2)e^{-9} + \ldots = \frac{\pi^2}{4} + 2e^{-1} - \frac{2}{3}e^{-9} + \ldots$$

Since $e^{-9} \ll e^{-1}$, we can ignore that term, as well as later terms. So,

$$\boxed{u(\pi/2,1) \approx \frac{\pi^2}{4} + 2e^{-1}}.$$

(Using a calculator this is approximately 3.203.)

End of quiz solutions