ES.1803 Problem Section Problems for Quiz 6, Spring 2024

Topic 24 Linear ODEs with periodic input

Problem 24.1. Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$. (You can use the known series for $f(t)$.)

Problem 24.2. **(a)** Solve $x'' + 2x' + 9x = g(t)$, where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$. Find the Fourier series of g by using $g(t) = f(\pi t)/\pi$, where f is the standard period 2π triangle wave $f(t) = |t|$ on $[-\pi, \pi]$.

(b) Is there a term in the Fourier series for g whose frequency is near the natural frequency of the system modeled by the DE? For the response found in Part (a), does this term have the largest amplitude?

Problem 24.3. Solve $x'' + 4x = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$. Look out for resonance.

Topic 25 PDEs: heat and wave equations

Problem 25.4. **(Linearity)** Assume we have a heated rod of length L with its ends in ice baths. We can model this using the heat equation with boundary conditions.

For functions $u = u(x,t)$, the PDE

$$\frac{\partial u}{\partial t}(x,t) = a\frac{\partial^2 u(x,t)}{\partial^2 x}$$

is the heat equation. In this problem we want to look at linearity of this equation and also of boundary conditions.

(a) The PDE can be written as $\left(\dfrac{\partial}{\partial t} - a\dfrac{\partial^2}{\partial x^2}\right) u = 0.$

We can use the language of operators: The partial differential operator $\mathcal{T} = \left(\dfrac{\partial}{\partial t} - a\dfrac{\partial^2}{\partial x^2}\right)$ is called the **heat operator**. The heat equation is simply

$$\mathcal{T}u = 0.$$

Show the heat operator is linear.

(b) Show the heat equation $\mathcal{T}u = 0$ is homogeneous. That is, if u_1 and u_2 are solutions then so are $c_1u_1 + c_2u_2$.

(c) The boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ also have solutions, i.e., functions that satisfy the boundary conditions.

Show the boundary conditions are linear and homogeneous. That is, we can superposition solutions and get solutions.

(d) Show that the combined system of the heat equation plus the given boundary conditions is linear and homogeneous.

Problem 25.5. Consider the following heat equation with boundary conditions.

PDE: $u_t(x,t) = 4u_{xx}(x,t)$, for $0 \le x \le \pi$, $0 \le t$.

BC: $u(0,t) = 0$, $u(\pi,t) = 0$.

(a) Find the general solution.

(b) Now consider the initial condition (you should graph this).

(IC) $u(x,0) = f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi/2 \\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$

Find the solution to the PDE that satisfies both the BC and the IC.

(c) If this models the temperature of a heated rod, what happens to the temperature over time? Which mode is the dominant mode?

Problem 25.6. Solve the wave equation with boundary and initial equations.

PDE: $y_{tt} = y_{xx}$ for $0 \le x \le 1$, $t > 0$.

BC: $y(0,t) = 0$, $y(1,t) = 0$

IC: $y(x,0) = 0$, $y_t(x,0) = 1$.

Problem 25.7. (This problem uses a cosine series, so the $\lambda = 0$ case is important.)

(a) Solve the heat equation with insulated ends.

PDE: $u_t = 3u_{xx}$ for $0 \le x \le 1$, $t > 0$.

BC: $u_x(0,t) = 0$, $u_x(1,t) = 0$

IC: $u(x,0) = x$.

(b) Write out explicitly (compute values of coefficients) the first 4 nonzero terms when $t = 1/32$, i.e., write the first four terms of $u(x, 1/32)$. Use this to explain why, after a very short time, the constant and $n = 1$ term give a very good approximation of the solution.

Topic 26: Continuation; applications to sound.

Problem 26.8. **(a)** Find the general solution to the following heat equation with inhomogeneous boundary conditions

PDE: $u_t(x,t) = 4u_{xx}(x,t)$, for $0 \le x \le \pi$, $0 \le t$.

BC: $u(0,t) = 1$, $u(\pi,t) = 2$.

This has inhomogeneous boundary conditions. So we will use the strategy of finding a particular solution to the above and adding the general solution to the associated homogeneous equation. The homogeneous equation is a Topic 25 problem. Here is the solution:

$$u_h(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin(nx)$$

(If you haven't solved that problem yet, you should do that now.)

(b) Find the solution that also satisfies the initial condition $u(x, 0) = 2$.
